18.02SC Multivariable Calculus, Fall 2010
Transcript – Recitation 43, Non-Conservative Vector Fields


CHRISTINE BREINER: Welcome back to recitation. In this video, I want us to work on the following problem, which is to show that this vector field is not conservative. And the vector field is minus yi plus xj all divided by x squared plus y squared. So you can think about this in two separate components, if you need to, as minus y divided by x squared plus y squared i plus x over x squared plus y squared j. So that's really exactly the same thing. So your object is to show that this vector field is not conservative. And why don't you work on that for awhile, pause the video, and then when you're ready to see my solution you can bring the video back up.


So welcome back. Well, maybe some of you thought I had a typo in this problem initially, and I wanted you to show it , in fact, was conservative, but it actually is not a conservative vector field. And let me explain how we can show it is not conservative and why probably what you did initially to try and show it was not-- didn't work. So maybe that wording was a little confusing, but let me take you through it.


So the first thing I would imagine you tried is you looked at m sub y and n sub x. So m in this case is negative y over x squared plus y squared. And N in this case, capital N in this case, is x divided by x squared plus y squared. So if you worked that out, you probably did or maybe you didn't, and I'll just show you. M sub y-- let me just double check-- is y squared minus x squared over x squared plus y squared, I think with an extra squared on it. Yeah. And that's also equal to N sub x. Right?


So what you know so far, what you might have thought immediately, was well, N sub x minus M sub y is the curl of F and that's equal to 0, and therefore this vector field is conservative. But the problem is the theorem you were thinking about referencing doesn't hold. And the reason is because there are two hypotheses in that theorem. And one is that if I define this vector field, if I call it capital F, the vector field-- or the theorem is that capital F defined everywhere, and curl of F equal to 0, implies F conservative.


OK. So that's the theorem you might have been trying to use. You see from this the curl of F equals 0, but the problem is the first part of this statement, that F being defined everywhere, is not true. In fact, there's one place in r2 where this vector field is not defined, and that is when x is 0 and y is 0. Because at that point, obviously, the denominator is zero and we run into trouble. So you cannot use this theorem to say F is conservative because it's not defined everywhere. Or I should be careful how I say that. There is somewhere that it is not defined. So even though the curl of F equals 0, the first part of the statement

is not true. So you cannot get anything out of this theorem. So knowing the curl of F equals 0 doesn't tell you whether it's conservative or not. OK?

So now what I'm going to do is I'm going to show-- I told you we want to show it's not conservative. I'm going to show you how we can show that. And what we're going to do is we're going to find a loop, a closed loop. So a closed curve in R2 that when I integrate this vector field over that closed curve, I don't get zero. And then we would know that the vector field is not conservative. So that's what we're going to do. So let me write it out explicitly and then we'll figure out the curve we want and then we'll parameterize the curve appropriately.

So I'm going to show for some closed curve. I'm going to pick my curve and I'm going to integrate over the closed curve and I'm going to integrate this. Minus y over x squared plus y squared dx plus x over x squared plus y squared dy. And I'm going to show that if I pick a certain curve, I'm going to get something that's not equal to zero, OK? And the curve I'm going to pick is the unit circle. So we're going to let C be the unit circle. Let C equal-- I'll just write the unit circle.

But how can I parameterize the unit circle easily? I can parameterize the unit circle easily by x equal to cosine theta and y equal to sine theta. So let me do that. And why am I picking the unit circle? We'll see why that is in a second. So we're going to let x equal cosine theta and y equal sine theta. And so now we know what goes here and we know what goes here. By the way, what is x squared plus y squared? It's cosine squared theta plus sine squared theta, which is equal to 1. This is exactly the square of the radius and since we're on the unit circle, that's 1. That's why I picked the unit circle because I wanted the denominator to be very simple.

So I've got the x's and the y's. Now what's dx? dx is going to be equal to minus sine theta d theta. And what's dy? I'll just write it right underneath here. dy is going to equal cosine theta d theta. So let me just point out again what we were doing here. We want to parameterize the unit circle. I chose to parameterize it in theta, which I haven't told you what my bounds are yet, but I've done everything else. I needed to know what x and y were and also what dx and dy were. And so now I can start substituting in.

So let's figure out what I get when I start substituting in. I'm integrating now. Again, I said I didn't mention the bounds. What are the bounds on theta to get the whole unit circle? I'm just going to integrate from 0 to 2 pi. So I integrate from 0 to 2 pi. That takes me all the way around the circle. Minus y is negative sine theta. This part is 1 as I mentioned earlier. And then dx is minus sine theta d theta. So I

have a minus sine theta times a minus sine theta. That's going to give me a sine squared theta d theta. And then this term, the second term, when I integrate over the curve, I'm going to just rewrite another one here separately momentarily. x is cosine theta. x squared plus y squared is 1. And dy is cosine theta d theta. So I get cosine squared theta d theta. All right?

Now here we are. If I tried to integrate both of these separately, it would take potentially a very long time and be kind of annoying. But if you notice, because I can add these integrals, I can add over they have the same bounds, so I can put them back together. Sine squared theta plus cosine squared theta always equals 1. That's a trig identity that's good to know. So this in fact is equal to the integral from 0 to 2 pi of d theta.

So let me come back one more time and make sure we understand. We're integrating from 0 to 2 pi sine squared theta d theta plus cosine squared theta d theta. That's just sine squared theta plus cosine squared theta d theta. So that's 1. So 1 times d theta. But what is this? Well, this integral from 0 to 2 pi of d theta is theta evaluated at 2 pi and 0. I actually get 2 pi, which is in particular not equal to 0, right? So that actually shows you that this vector field is not conservative.

Now why does this make sense? This makes sense because if I really think about what I'm doing-- actually, this is the place where maybe it'll ultimately make the most sense-- what you're doing is you're looking at how theta changes as you go all the way around the origin once. And theta changes by 2 pi. If you go all the way around one time, the theta value starts at 0 and then goes up to 2 pi. And so that's exactly where this 2 pi is coming from. So that actually is ultimately how you were going to be able to show that this vector field is not conservative.

So let me go back and just remind you where we came from. We started off with a vector field and we wanted to know if it was not conservative. We wanted to show-- sorry-- it was not conservative. So the first thing you might want to check, which maybe you did, was if the curl was zero. And in fact the curl is zero. And so maybe you thought, well, she might have done something wrong, or she might have written something wrong. But we can't actually appeal to the theorem you want to appeal to to make any conclusion about the vector field because the vector field in our example is not defined for every value of (x,y). So for every value in the xy-plane, we cannot define F. There's one value for which we cannot define F. And so we cannot say that if the curl's 0, then the vector field is conservative. We can't draw any conclusions from this theorem.

So then we had to actually find a way to show it was not conservative without looking at the curl. And that amounts to showing there is a closed curve that when I integrate over that closed curve-- when I look at what the vector field does over that closed curve-- I get something non-zero. And we picked an easy example. This is actually what the integral will look like over the closed curve in x and y. We let our closed curve be the unit circle, then we parameterized in theta. And we see that actually what this vector field is doing is it's looking at what is d theta? It's finding out what d theta is. And so we find out the integral from 0 to 2 pi of d theta is obviously not zero. And that gives us that the vector field's not conservative. So that's where I'll stop.

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