

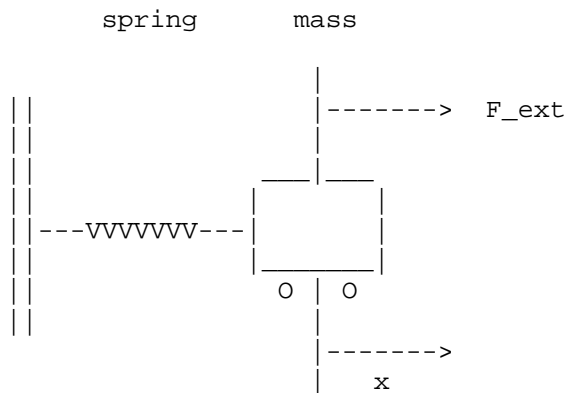
18.03 Class 11, Feb 26, 2010

Second order linear equations:  
Physical model, solutions in homogeneous case.  
Characteristic polynomial, distinct real roots.

- [1] Springs and masses
- [2] Dashpots
- [3] Second order linear equations
- [4] Solutions in homogeneous case: Superposition I
- [5] Exponential solutions: characteristic polynomial

[1] Second order equations are the basis of analysis of mechanical and electrical systems. We'll build this up slowly.

A spring is attached to a wall and a cart:



Set up the coordinate system so that at  $x = 0$  the spring is relaxed, which means that it is exerting no force.

In addition to the spring, suppose that there is another force acting on the cart -- an "external force," maybe wind blowing on a sail attached to it, maybe gravity, or some other force. Then

$$mx'' = F_{\text{spr}} + F_{\text{ext}}$$

The spring force is characterized by the fact that it depends only on position. In fact:

$$\begin{aligned} \text{If } x > 0 &, F_{\text{spr}}(x) < 0 \\ \text{If } x = 0 &, F_{\text{spr}}(x) = 0 \\ \text{If } x < 0 &, F_{\text{spr}}(x) > 0 \end{aligned}$$

I sketched a graph of  $F_{\text{spr}}(x)$  as a function of  $x$ . The simplest way to model this behavior (and one which is valid in general for small  $x$ , by the tangent line approximation) is

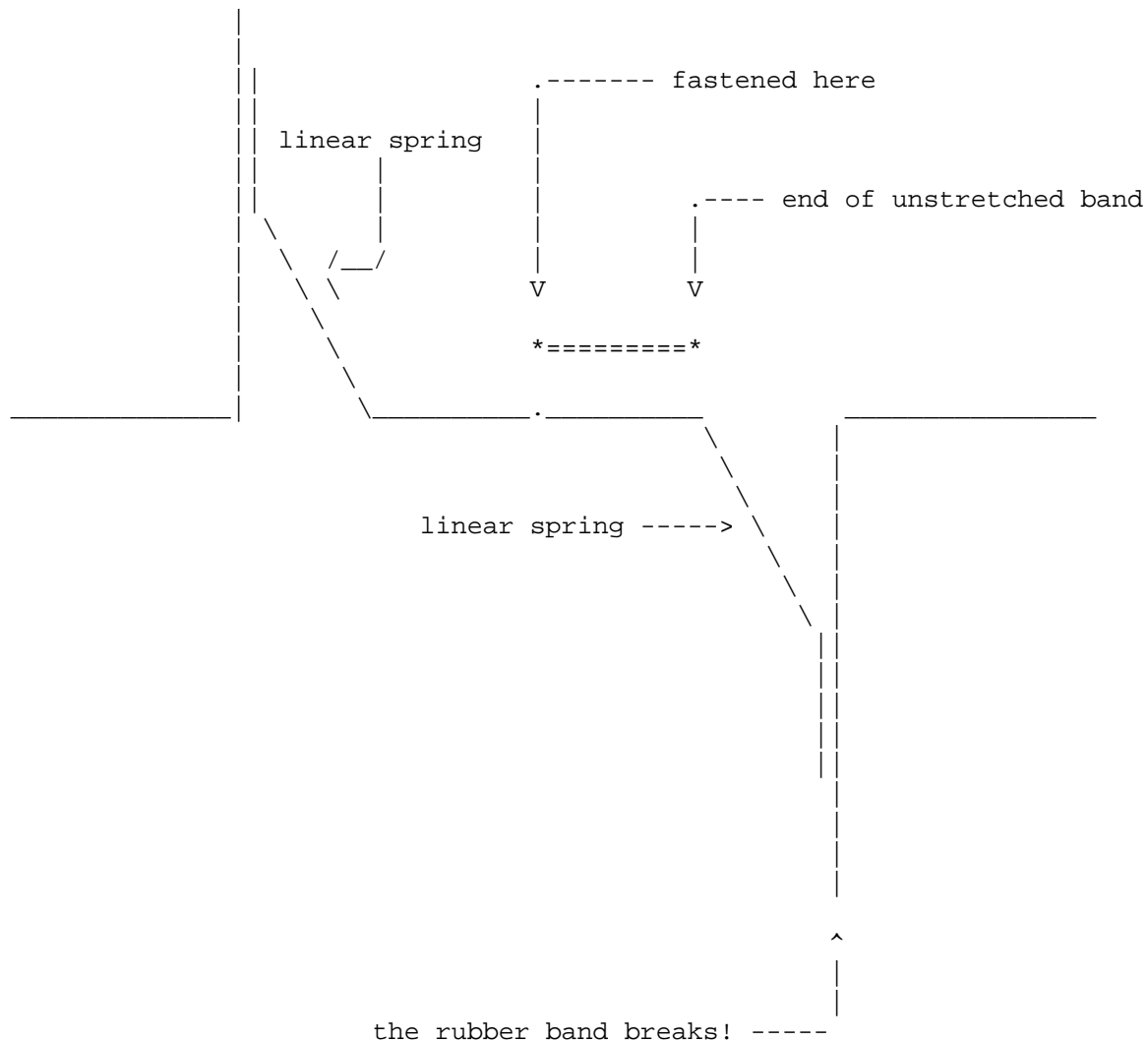
$$F_{\text{spr}}(x) = -kx \quad k > 0 \quad \text{the "spring constant."} \quad \text{"Hooke's Law"}$$

This is another example of the linear approximation that Linn was

discussing on Monday. So we get

$$m\ddot{x} + kx = F_{\text{ext}} .$$

I displayed a weight on a rubber band. This is not a spring, as you usually think of one, but it behaves like one, at least in a range. Lay a rubber band laid out on a table. Fix the right end of it and set  $x = 0$  where the left end is in a relaxed state, then the graph of the force exerted by the rubber band looks something like this ...



[2] Any real mechanical system has friction. Friction takes many forms. It is characterized by the fact that it depends on the motion of the mass. We will suppose that it depends only on the velocity of the mass and not on its position. Often the damping is controlled by a "dashpot." This is a cylinder filled with oil, that a piston moves through. Door dampers and car shock absorbers often actually work this way. Write  $F_{\text{dash}}(x')$ .

It acts against the velocity:

If  $x' > 0$  ,  $F_{\text{Dash}}(x') < 0$   
 If  $x' = 0$  ,  $F_{\text{Dash}}(x') = 0$   
 If  $x' < 0$  ,  $F_{\text{Dash}}(x') > 0$

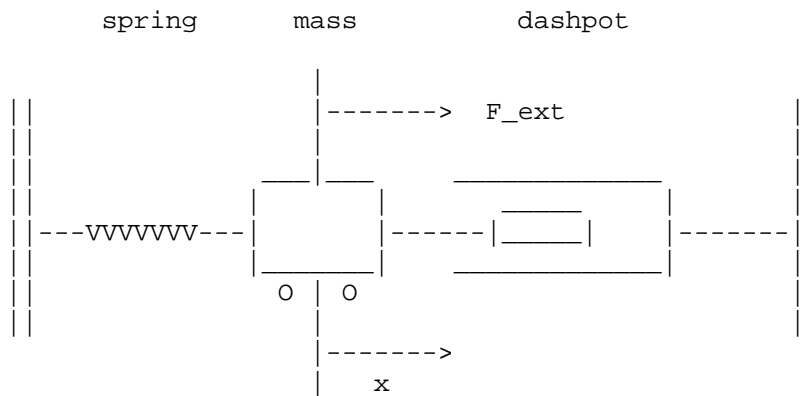
The simplest way to model this behavior (and one which is valid in general for small  $x'$  , by the tangent line approximation) is

$$F_{\text{fric}}(x) = -bx \quad b > 0 \quad \text{the "damping constant."}$$

This is "linear damping." Altogether the equation is

$$mx'' + bx' + kx = F_{\text{ext}}$$

Diagrammatically:



[3] A linear differential equation is one of the following form:

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_1 x' + a_0 = q(t)$$

(no constant term on the left hand side!) The  $a_k$ 's are the "coefficients." They may depend upon  $t$  (but not on  $x$  ). If  $a_n$  is not zero, this is "of order  $n$ ."

Our spring system is an example of a \*second order\* linear equation. (Two springs in series will give a fourth order equation.)

The left hand side represents the SYSTEM, the spring/mass/dashpot system. The right hand side encodes the INPUT SIGNAL, an external force at work. The coefficients represent parameters of the system -- for example, mass, damping constant, spring constant. In general they may depend upon time: maybe the force is actually a rocket, and the fuel burns so  $m$  decreases. Maybe the spring gets softer as it ages. Maybe the honey in the dashpot gets stiffer with time.

Most of the time we will assume that the coefficients are CONSTANT: the timescale of their variation is much longer than the timescale of the dynamical variable  $x$  . But the external force  $F_{\text{ext}}(t)$  can certainly vary (maybe sinusoidally).

We can see physically that there are many solutions with fixed

initial value of  $x$  ,  $x(t_0) = x_0$  . So many different solution curves of a second order equation pass through a given point. But, also on physical grounds, the initial position together with the initial velocity determine the solution: "initial position" data for a second order equation consist of both position and velocity at a give instant. This is a version of the existence/uniqueness theorem. In an  $n$ -th order equation, you need to know  $n$  numbers:  $x(t_0), x'(t_0), \dots x^{(n-1)}(t_0)$  .

[4] We'll begin with  $F_{\text{ext}} = 0$  : the system is allowed to evolve on its own, without outside interference.

$$mx'' + bx' + kx = 0 \quad , \quad m \text{ nonzero} \quad (*)$$

Think of a saloon door swinging. This is a \*homogeneous\* equation.

Special case (from recitation): the harmonic oscillator:  $b = 0, k > 0$  :

$$x'' + (k/m) x = 0$$

Solutions come from simple facts about sines and cosines:

$$\begin{aligned} \cos(\omega t) & \text{ ---> } -\omega \sin(\omega t) & \text{ ---> } -\omega^2 \cos(\omega t) \\ \sin(\omega t) & \text{ ---> } \omega \cos(\omega t) & \text{ ---> } -\omega^2 \sin(\omega t) \end{aligned}$$

So if  $\omega^2 = k/m$  , so that our equation is  $x'' + \omega^2 x = 0$  ,

then  $x = \cos(\omega t)$  and  $x = \sin(\omega t)$  are solutions:

ANOTHER FUNDAMENTAL FACT TO MEMORIZE!

In fact, you showed that any sinusoid of circular frequency  $\omega$ ,

$$x = a \cos(\omega t) + b \sin(\omega t) = A \cos(\omega t - \phi) \quad (*)$$

is also a solution. In fact these are the only solutions, because

$$x(0) = a$$

$$x'(0) = \omega b$$

and so you can solve (uniquely) for  $a$  and  $b$  to give any desired initial condition: you don't need more.

Q11.1. What is the period of a nonzero solution of  $x'' + 4x = 0$  ?

1. Depends upon the solution
2. 2
3.  $\pi$
4. 4
5.  $2\pi$
6.  $\pi/2$

Blank. Don't know.

Ans: : think of what t has to do to take (2t) from 0 to 2pi.  
 Or use  $P = 2\pi/\omega$ , with  $\omega = 2$ . Ans: 3: pi.

The fact that  $\cos(\omega t)$  and  $\sin(\omega t)$  are solutions implies that (\*) is also a solution, via

Superposition I: If  $x_1$  and  $x_2$  are two solutions of the homogeneous linear equation, then so is

$$x = c_1 x_1 + c_2 x_2$$

Check (in second order case  $mx'' + bx' + kx = 0$ ):

$$\begin{aligned} & m(c_1 x_1 + c_2 x_2)'' + b(c_1 x_1 + c_2 x_2)' + k(c_1 x_1 + c_2 x_2) \\ &= m(c_1 x_1'' + c_2 x_2'') + b(c_1 x_1' + c_2 x_2') + k(c_1 x_1 + c_2 x_2) \\ &= c_1(m x_1'' + b x_1' + k x_1) + c_2(m x_2'' + b x_2' + k x_2) \\ &= c_1 (0) + c_2 (0) = 0. \end{aligned}$$

[5] The equation  $mx'' + bx' + kx = 0$ , for  $m, b, k$  constant, is a lot like  $x' + kx = 0$ , which has as solution  $x = e^{-kt}$  (and more generally multiples of this). It makes sense to try for exponential solutions of (\*):  $e^{rt}$  for some as yet undetermined constant and  $r$ .

To see which  $r$  might work, plug  $x = e^{rt}$  into (\*). Organize the calculation: the  $k$ ,  $b$ ,  $m$  are flags indicating that I should multiply the corresponding line by this number.

$$\begin{array}{l} k ] \quad x = c e^{rt} \\ b ] \quad x' = c r e^{rt} \\ m ] \quad x'' = c r^2 e^{rt} \end{array}$$

---


$$0 = mx'' + bx' + kx = c (mr^2 + br + k) e^{rt}$$

An exponential is never zero, so we can cancel to see that  $c e^{rt}$  is a solution to (\*) for any  $c$  exactly when  $r$  is a root of the "characteristic polynomial"

$$p(s) = ms^2 + bs + k$$

Example.  $x'' + 5x' + 4x = 0$

The characteristic polynomial  $s^2 + 5s + 4$ . We want the roots. One reason I wanted to write out the polynomial was to remember that you can find roots by factoring it. This one factors as  $(s + 1)(s + 4)$  so the roots are  $r = -1$  and  $r = -4$ . The corresponding exponential solutions are  $e^{-t}$  and  $e^{-4t}$ .

By superposition, the "linear combination"

$$x(t) = c_1 e^{-t} + c_2 e^{-4t}$$

is a solution as well. This is the general solution.

Suppose we know also that  $x(0) = 2$  and  $x'(0) = -5$ . To use the second condition we'll need to know

$$x(t) = -c_1 e^{-t} - 4c_2 e^{-4t}$$

Then

$$\begin{array}{rcl} 2 & = & c_1 + c_2 \\ -5 & = & -c_1 - 4c_2 \\ \hline -3 & = & -3c_2 \end{array} \quad \text{so } c_2 = 1 \text{ and then } c_1 = 1:$$

$$x(t) = e^{-t} + e^{-4t}$$

General picture: A linear equation of degree  $n$

$$a_n x^{(n)} + \dots + a_1 x' + a_0 x = q(t)$$

with constant coefficients has a characteristic polynomial,

$$p(s) = a_n s^n + \dots + a_1 s + a_0$$

$e^{rt}$  is a solution if and only if  $r$  is a root of  $p(s)$  :  $p(r) = 0$ .

By superposition, any linear combination of these exponentials is also a solution.

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