

18.03 Class 39, May 12, 2010

Dangers of linearization; limit cycles; chaos; next steps in mathematics

1. Stability
2. Limits of linearization
3. Limit cycles
4. Strange attractors
5. Essential skills
6. Next steps

[1] Let's say that an equilibrium is
-- "stable" if all nearby trajectories stay near to it and converge to
it as $t \rightarrow \infty$
-- "unstable" otherwise

We classified linear phase portraits using the (Tr, Det) plane.
In the (Tr, Det) plane, the stable region is the upper left quadrant.
It is bounded by the straight half-lines where either the eigenvalues
are purely imaginary, or one is zero and the other is negative.

[2] The method we sketched on Friday and Monday works well "generically,"
i.e. almost all the time. It lets you predict what the phase portrait of
an autonomous system looks like near equilibrium, most of the time,
and whether the equilibrium is stable or not.

The facts: (1) If the linearization is not on the borderline of the stable
quadrant, it correctly predicts the stability of the equilibrium.

(2) If the eigenvalues of the Jacobian have nonzero real parts - so
 (tr, det) is neither on the horizontal axis nor on the $\text{tr} = 0, \text{det} > 0$ line --
then the phase portrait is a small deformation of the linear phase portrait.

Example: $x' = -y - y^2, y' = x + y^2$.
The linearization at $(0,0)$ is $x' = y, y' = -x$: $\text{trace} = 0, \text{det} > 0$,
so you expect centers (which are not stable), but the fact is that you get
stable spirals (seen on a pplane plot). Other examples might show
stable spirals instead, or centers. You can't tell which you get from the
linearization alone. [These spiral decay to zero more slowly than exponential,
though.]

Example: $x' = 4xy, y' = x^2 - y^2$.
The linearization at $(0,0)$ is now $x' = 0, y' = 0$:

$$J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Degenerate: $\text{det} = 0$. In fact *both* eigenvalues are zero (and it is
complete). What is the phase portrait of this linear system? It just says
 $u' = 0$ so every point is a constant solution. The eigenvalues are both zero,
every vector is an eigenvector, every solution is constant, it is a
degenerate system.

On the other hand, a Matlab plot [Slide] of the phase portrait shows that
it has six ray solutions, unlike any linear portrait. Weird. Second order
effects dominate.

[3] In studying a nonlinear autonomous system, the most important thing, after equilibria, is the possibility of periodic solutions.

For example, the "van der Pol" equation is

$$x'' + a(x^2 - 1)x' + x = 0$$

for fixed a . When $a = 0$ this is just the harmonic oscillator, with $\omega_n = 1$; all the nonzero solutions are periodic of period 2π .

The companion system is

$$\begin{aligned}x' &= y \\y' &= -x - a(x^2 - 1)y\end{aligned}$$

This system turns out to continue to have periodic solutions. When $a > 0$ the situation is in fact even better: there is ONLY ONE periodic trajectory, and all other nonzero solutions converge to it. This is a "limit cycle." The human heart is in fact controlled by an equation like this. This is why it dependably returns to a normal periodic pattern after being disturbed.

The limit cycle is an "attractor," in the sense that every nearby trajectory stays nearby and converges to it. I demonstrated the Mathlet "Vector Fields."

[4] Limit cycles are typical for 2D systems.

But when you move to 3D things are much more complicated. The first such system was discovered right here at MIT by Edward Lorenz [Slide], who was modeling "convection rolls" in the upper atmosphere. In 1963 he wrote down a fairly simple model, a nonlinear autonomous system in 3 dimensions:

$$\begin{aligned}x' &= -ax + by \\y' &= -xz + rx - y \\z' &= xy - bz\end{aligned}$$

for constants a, b, r . Here's what happens for certain values of these parameters - and I showed the IDE tool "Lorenz Equations, Phase Plane, $0 < r < 30$ ".
<<http://www.aw-bc.com/ide/idefiles/media/JavaTools/lrnzr320.html>>.

The solutions don't ever settle down to a periodic orbit; but neither do they run off to infinity. They just wrap pretty crazily around the two nonzero unstable equilibria which exist provided that $r > a/b$.

Now the "attractor" is much stranger than just a loop, and, strangely, is called a "strange attractor." This is the fundamental characteristic of "chaos."

[5] In addition to these skills [Slide] there are a couple of big ideas: Linearity is one. The exponential function is another.

1. First order models. Euler's method. Autonomous equations.
2. First order linear equations: Integrating factors
3. Complex numbers and exponentials.
4. Second order LTI systems: Superposition; Exponential Response Formula; exponential or sinusoidal signal; amplitude gain and phase lag; Polynomial signal; Variation of parameter.
5. Delta functions, unit impulse response, convolution.
6. Fourier series, periodic solutions.
7. Laplace transform; transfer function and weight function.
8. Linear systems: eigenvalues, eigenvectors.
9. Linear phase portraits.
10. Nonlinear phase portraits; linearization at equilibria.

[6] Study resources:

Practice final.

OCW Final Exam and Practice Final.

Absent from this year's curriculum:

--variation of parameters as an alternative to integrating factors

--``Exponential Shift Law'' (We did these problems using a different
 ``variation of parameters: solve $p(D)x=q(t)e^{rt}$ by trying $x=u(t)e^{rt}$
 and finding a differential equation for $u(t)$.)

--Laplace transform solution of second order equations with non-rest initial conditions

Other resources:

Hour exams and practice hour exams

Recitation problems and their solutions

Homework and solutions

MIT OpenCourseWare
<http://ocw.mit.edu>

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