Recitation 14, March 30, 2010

Fourier Series

1. Graph the function $f(t)$ which is even, periodic of period 2π, and such that $f(t) = 2$ for $0 < t < \frac{\pi}{2}$ and $f(t) = 0$ for $\frac{\pi}{2} < t < \pi$. Find its Fourier series in two ways:

(a) Use the integral expressions for the Fourier coefficients. (Is the function even or odd? What can you say right off about the coefficients?)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 2dt = 2.$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2} 0dt + \int_{-\pi/2}^{\pi/2} 2\cos(nt)dt + \int_{\pi/2}^{\pi} 0dt\right) \\ &= \frac{2}{n\pi}\sin(nt)|_{-\pi/2}^{\pi/2} \\ &= \frac{4}{n\pi}\sin(n\pi/2) \end{aligned}$$

If n is even, this is zero. If n is odd, it is $\frac{4}{n\pi}\sin(n\pi/2) = \pm\frac{4}{n\pi}$. More specifically, if $n-1$ is a multiple of four, then it is the positive option $\frac{4}{n\pi}$, and if $n-3$ is a multiple of four, then we get the negative option $-\frac{4}{n\pi}$.

The function $f(t)$ is even, so $b_n = 0$ for all $n > 0$. We can verify it as the following:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2} 0dt + \int_{-\pi/2}^{\pi/2} 2\sin(nt)dt + \int_{\pi/2}^{\pi} 0dt\right) \\ &= -\frac{2}{n\pi}\cos(nt)|_{-\pi/2}^{\pi/2} \\ &= 0 \end{aligned}$$

The last equality is because cosine is an even function. The Fourier series is then

$$f(t) = 1 + \frac{4}{\pi}\cos t - \frac{4}{3\pi}\cos(3t) + \frac{4}{5\pi}\cos(5t) - \frac{4}{7\pi}\cos(7t) + \dots$$

(b) Express $f(t)$ in terms of $\mathrm{sq}(t)$, substitute the Fourier series for $\mathrm{sq}(t)$, and use some trig id.

$f(t) = 1 + \text{sq}(t + \pi/2)$. Since the Fourier series for $\text{sq}(t)$ is

$$sq(t) = \frac{4}{\pi}(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \dots)$$

then we can substitute to get

$$\begin{aligned} f(t) &= 1 + \frac{4}{\pi}(\sin(t + \pi/2) + \frac{1}{3}\sin(3t + 3\pi/2) + \frac{1}{5}\sin(5t + 5\pi/2) + \dots) \\ &= 1 + \frac{4}{\pi}\cos t - \frac{4}{3\pi}\cos(3t) + \frac{4}{5\pi}\cos(5t) - \dots \end{aligned}$$

(c) Now find the Fourier series for $f(t) - 1$.

$f(t) - 1$ is given by subtracting 1 from a_0, so we get

$$f(t) - 1 = \frac{4}{\pi}\cos t - \frac{4}{3\pi}\cos(3t) + \frac{4}{5\pi}\cos(5t) - \frac{4}{7\pi}\cos(7t) + \dots$$

2. What is the Fourier series for $\sin^2 t$?

$\sin^2 t$ is even, so $b_n = 0$ for all $n > 0$. Since $\cos(2t) = 1 - 2\sin^2 t$, $\sin^2 t = \frac{1}{2} - \frac{1}{2}\cos(2t)$. That is, $a_0 = 1$, $a_2 = -1/2$, and every other coefficient is zero.

3. Graph the odd function $g(x)$ which is periodic of period π and such that $g(x) = 1$ for $0 < x < \frac{\pi}{2}$. 2π is also a period of $g(x)$, so it has a Fourier series as above. Find it by expressing $g(x)$ in terms of the standard squarewave.

One can see that $g(t) = \text{sq}(2t)$, so the Fourier series can be found by doubling the frequencies associated to the coefficients.

$$g(t) = \frac{4}{\pi}\sin(2t) + \frac{4}{3\pi}\sin(6t) + \frac{4}{5\pi}\sin(10t) + \frac{4}{7\pi}\sin(14t) + \dots$$

4. Graph the function $h(t)$ which is odd and periodic of period 2π and such that $h(t) = t$ for $0 < t < \frac{\pi}{2}$ and $h(t) = \pi - t$ for $\frac{\pi}{2} < t < \pi$. Find its Fourier series, starting with your solution to **1(c)**.

This is a triangular wave. It increases with slope 1 between $-\pi/2$ and $\pi/2$, and decreases with slope -1 between $-\pi$ and $-\pi/2$ and between $\pi/2$ and π. In other words, its derivative is $f(t) - 1$, so our answer is a constant times the integral of the Fourier expansion of $f(t) - 1$.

Since the function is odd, $a_0 = 0$. The rest of the terms are given by dividing:

$$h(t) = \frac{4}{\pi}\sin t - \frac{4}{9\pi}\sin(3t) + \frac{4}{25\pi}\sin(5t) - \frac{4}{49\pi}\sin(7t) + \dots$$

5. Explain why any function $g(x)$ is a sum of an even function and an odd function in just one way. What is the even part of e^x? What is the odd part?

We can make an even function from $g(x)$ by taking the sum $g(x)+g(-x)$. Similarly, we can make an odd function by subtracting: $g(x)-g(-x)$. Adding them yields $2g(x)$, so we go back and divide by two:

$$g(x)=\frac{g(x)+g(-x)}{2}+\frac{g(x)-g(-x)}{2}$$

To show that this decomposition is unique, we suppose we have another pair $h_{even}(x)+h_{odd}(x)=g(x)$, where $h_{even}(x)$ is even and $h_{odd}(x)$ is odd. Then $h_{even}(x)-\frac{g(x)+g(-x)}{2}$ is even, $-h_{odd}+\frac{g(x)-g(-x)}{2}$ is odd, and one can easily check from the assumption $h_{even}(x)+h_{odd}(x)=g(x)$ that $h_{even}(x)-\frac{g(x)+g(-x)}{2}=-h_{odd}+\frac{g(x)-g(-x)}{2}$.So $h_{even}(x)-\frac{g(x)+g(-x)}{2}$ is a function that is both even and odd, and it has to be zero. Thus, $h_{even}(x)=\frac{g(x)+g(-x)}{2}$ and $h_{odd}(x)=\frac{g(x)-g(-x)}{2}$.

The even part of e^x is $\frac{e^x+e^{-x}}{2}=\cosh x$. The odd part of e^x is $\frac{e^x-e^{-x}}{2}=\sinh x$.