19. Convolution

19.1. **Superposition of infinitesimals: the convolution integral.** The system response of an LTI system to a general signal can be reconstructed explicitly from the unit impulse response.

To see how this works, start with an LTI system represented by a linear differential operator L with constant coefficients. The system response to a signal $f(t)$ is the solution to $Lx = f(t)$, subject to some specified initial conditions. To make things uniform it is common to specify "rest" initial conditions: $x(t) = 0$ for $t < 0$.

We will approach this general problem by decomposing the signal into small packets. This means we partition time into intervals of length say Δt: $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t$, and generally $t_k = k\Delta t$. The kth signal packet is the null signal (i.e. has value zero) except between $t = t_k$ and $t = t_{k+1}$, where it coincides with $f(t)$. Write $f_k(t)$ for the kth packet. Then $f(t)$ is the sum of the $f_k(t)$'s.

Now by superposition the system response (with rest initial conditions) to $f(t)$ is the sum of the system responses to the $f_k(t)$'s separately.

The next step is to estimate the system response to a single packet, say $f_k(t)$. Since $f_k(t)$ is concentrated entirely in a small neighborhood of t_k, it is well approximated as a rate by a multiple of the delta function concentrated at t_k, $\delta(t - t_k)$. The multiple should be chosen so that the cumulative totals match up; that is, it should be the integral under the graph of $f_k(t)$, which is itself well approximated by $f(t_k)\Delta t$. Thus we replace $f_k(t)$ by

$$f(t_k) \cdot \Delta t \cdot \delta(t - t_k).$$

The system response to this signal, a multiple of a shift of the unit impulse, is the same multiple of the same shift of the weight function (= unit impulse response):

$$f(t_k) \cdot \Delta t \cdot w(t - t_k).$$

By superposition, adding up these packet responses over the packets which occur before the given time t gives the system response to the signal $f(t)$ at time t. As $\Delta t \to 0$ this sum approximates an integral taken over time between time zero and time t. Since the symbol t is already in use, we need to use a different symbol for the variable in the integral; let's use the Greek equivalent of t, τ ("tau"). The t_k's get replaced by τ in the integral, and Δt by $d\tau$:

$$x(t) = \int_0^t f(\tau)w(t-\tau)\,d\tau \tag{1}$$

This is a really wonderful formula. Edwards and Penney call it "Duhamel's principle," but they seem somewhat isolated in this. Perhaps a better name would be the "superposition integral," since it is no more and no less than an integral expression of the principle of superposition. It is commonly called the **convolution integral**. It describes the solution to a general LTI equation $Lx = f(t)$ subject to rest initial conditions, in terms of the unit impulse response $w(t)$. Note that in evaluating this integral τ is always less than t, so we never encounter the part of $w(t)$ where it is zero.

19.2. **Example: the build up of a pollutant in a lake.** Every good formula deserves a particularly illuminating example, and perhaps the following will serve for the convolution integral. It is illustrated by the Mathlet `Convolution: Accumulation`. We have a lake, and a pollutant is being dumped into it, at a certain variable rate $f(t)$. This pollutant degrades over time, exponentially. If the lake begins at time zero with no pollutant, how much is in the lake at time $t > 0$?

The exponential decay is described as follows. If a quantity p of pollutant is dropped into the lake at time τ, then at a later time t it will have been reduced in amount to $pe^{-a(t-\tau)}$. The number a is the decay constant, and $t-\tau$ is the time elapsed. We apply this formula to the small drip of pollutant added between time τ and time $\tau + \Delta\tau$. The quantity is $p = f(\tau)\Delta\tau$ (remember, $f(t)$ is a *rate*; to get a *quantity* you must multiply by time), so at time t the this drip has been reduced to the quantity

$$e^{-a(t-\tau)}f(\tau)\Delta\tau$$

(assuming $t > \tau$; if $t < \tau$, this particular drip contributed zero). Now we add them up, starting at the initial time $\tau = 0$, and get the convolution integral (1), which here is

$$x(t) = \int_0^t f(\tau)e^{-a(t-\tau)}\,d\tau. \tag{2}$$

We found our way straight to the convolution integral, without ever mentioning differential equations. But we can also solve this problem by setting up a differential equation for $x(t)$. The amount of this chemical in the lake at time $t+\Delta t$ is the amount at time t, minus the fraction that decayed, plus the amount newly added:

$$x(t + \Delta t) = x(t) - ax(t)\Delta t + f(t)\Delta t$$

Forming the limit as $\Delta t \to 0$, we obtain

$$\dot{x} + ax = f(t), \qquad x(0) = 0. \tag{3}$$

We conclude that (2) gives us the solution with rest initial conditions.

An interesting case occurs if $a = 0$. Then the pollutant doesn't decay at all, and so it just builds up in the lake. At time t the total amount in the lake is just the total amount dumped in up to that time, namely

$$\int_0^t f(\tau)\, d\tau\,,$$

which is consistent with (2).

19.3. **Convolution as a "product".** The integral (1) is called the *convolution* of $w(t)$ and $f(t)$, and written using an asterisk:

$$w(t) * f(t) = \int_0^t w(t-\tau) f(\tau)\, d\tau, \qquad t > 0. \tag{4}$$

We have now fulfilled the promise we made at the beginning of Section 18: we can explicitly describe the system response, with rest initial conditions, to any input signal, if we know the system response to just one input signal, the unit impulse:

Theorem. The solution to an LTI equation $Lx = f(t)$, of any order, with rest initial conditions, is given by

$$x(t) = w(t) * f(t),$$

where $w(t)$ is the unit impulse response.

If L is an LTI differential operator, we should thus be able to reconstruct its characteristic polynomial $p(s)$ (so that $L = p(D)$) from its unit impulse response. This is one of the things the Laplace transform does for us; in fact, the Laplace transform of $w(t)$ is the reciprocal of $p(s)$: see Section 21.

The expression (4) can be interpreted formally by a process known as "flip and drag." It is illustrated in the Mathlet `Convolution: Flip and Drag`.