18.03 Class 35, May 3, 2010

Linear Phase Portraits: Eigenvalues Rule (usually)

1. Eigenvalues rule!
2. Trace-determinant plane
3. Marginal cases
4. Stability

[1] Phase portrait: this means the (x,y) plane (the "phase plane") with trajectories of solutions of u' = Au drawn on it (with direction of time indicated).

These homogeneous linear equations exhibit a nice variety of phase portraits, as shown by the Linear Phase Portraits Mathlets. An important fact is this:

EIGENVALUES RULE

We'll classify the linear phase portraits according to the eigenvalues of the matrix A .

Example: those rabbits again: A = [ .3 .1 ; .2 .4 ]

p_A(lambda) = lambda^2 - .7 lambda + .1

has roots lambda_1 = .5 , lambda_2 = .2

so we learn that all solutions flee from the origin: the phase portrait is a "source."

lambda = .5 : [ -.2 .1 ; .2 -.1 ] ==> v_1 = [ 1 ; 2 ]

so one normal mode is u_1 = e^{.5 t} [ 1 ; 2 ]

If the number of rabbits in MacGregor's field is twice the number in Jones's, it stays that way forever after.

lambda = .2 : [ .1 .1 ; .2 .2 ] ==> v_2 [ 1 ; -1 ]

so the other normal mode is u_2 = e^{.2t} [ 1 ; -1 ] .

This is not meaningful in itself in our population model, but we can draw it in the phase plane. And the two together provide the general solution.

What to the other trajectories look like?

For large t , the v_1 component is much bigger than the v_2 component.
For small t , the v_1 component is much smaller than the v_2 component.

So near the origin the trajectories become asymototic to the eigenline with smaller eigenvalue.

This phase portrait is a NODE. The same kind of picture occurs whenever the eigenvalues are real, of the same sign, and distinct.

[2] When are the eigenvalues non-real?

[Slide:]   p_A(lambda) = det ( A - lambda I )

If  A = [ a b ; c d ]  then   p_A(lambda) = lambda^2 - (tr A) lambda + (det A)

      tr(A) = a + d = lambda_1 + lambda_2

     det(A) = ad-bc = lambda_1 lambda)2

To find the eigenvalues, complete the square:

      p_A(lambda) = (lambda - (tr(A)/2))^2 + (det(A) - (tr(A)/2)^2)

so     lambda1,2  =  tr(A)/2 +- sqrt(tr(A)^2/4 - det(A)).

lambda1,2   are not real if  det(A) > tr(A)^2/4

            are equal if     det(A) = tr(A)^2/4

            are real and different from each other if  det(A) < tr(A)^2/4

The boundary is the "critical parabola," where det(A) = tr(A)^2/4.

Notice that if the eigenvalues are complex, the real part is  tr(A)/2.
If the eigenvalues are real, they have the same sign exactly when their
product is positive, and that sign is positive if their sum is also
positive.  Thus:

```
                         det

                          ^
    .                     |<-----purely imaginary
     .             complex e.v.s              .
      .                   |                  .
       .      Re < 0      |     Re > 0       .<----- repeated
         .            |                .
           .              |              .
   real < 0  .            |            .     real > 0
               ..         |         ..
                   ..     |     ..
  ------------        .......        ------------>  tr
                                         ^
                                         |
                 real, opposite sign     |______ at least one zero e.v.
```

The eigenvalues determine the general characteristics of the solutions.

NODES:  When the eigenvalues are real and of the same sign, but
distinct, you have a "node." If the sign is positive,
it's "unstable" or a "source." If negative, "stable" or a "sink."

SPIRALS: when the eigenvalues are non-real, we get spirals.
Two more comments on this:

(1) The spirals move IN when Re(lambda) < 0 "stable spiral"
OUT when Re(lambda) > 0 "unstable spiral"

When Re(lambda) = 0 it turns out that the trajectories are ellipses.
The technical term for this type of phase portrait is "center."

SADDLES: When the eigenvalues are real and of opposite sign, the phase plane is a "saddle." There are two eigenlines, one with positive eigenvalue and the other with negative. Normal modes along one move out, and along the other move in. The general solution is a combination of these two.

For example A = [ 1 1 ; 0 -2 ] has tr(A) = -1 , det(A) = -2
so the eigenvalues are lambda1 = 1 , lambda2 = -2

Eigenvector for lambda1 is again [1;0] , since the first column of the matrix is [1;0] .

For lambda2 we subtract -2 from the diagonal: [ 3 1 ; 0 0 ]
so v2 = [-1;3] .

The trajecories of other solutions are asymptotic to the eigenline for value +1 as t ---> infinity and to the eigenline for value -1 as t ---> - infinity.

The corresponding phase portraits exhibit the following behaviors:

```
  stars or               det
  defective nodes
     |                    ^
    .V                    |<-----centers        .
     .                    |                    .
      .    stable         |   unstable        .
       .   spirals        |   spirals        .
         .            |                .
           .              |              .
   stable    .            |            .   unstable
   nodes       ..         |         ..     nodes
                   ..     |     ..
  ------------         .......         ------------>  tr
                                             ^
                                             |
                       saddles               |______  degenerate cases
```

[3] Marginal cases: For the most part, the trace and determinant determine the name and essential features of the phase portrait, but there are some things which require inspecting the matrix itself.

-- det = tr^2/4 , along the critical parabola: repeated real eigenvalues.

For example on Friday [Slide] we discussed the matrix  A = [ -2 1 ; -1 0 ]
and saw that it has eigenvalues  lambda_1 = lambda_2 = -1 ,
eigenvector  v_1 = [ 1 ; 1 ] , so normal mode  u_1 = e^{-t} [ 1 ; 1 ]
Then we found a solution  w  to   (A - lambda-1) w = v_1  to be
w = [ 0 ; 1 ] , and wrote down the extra solution

u_2 = e^{lambda_1 t} ( t v_1 + w ) = e^{-t} [ t [ 1 ; 1 ] + [ 1 ; 0 ] )

= e^{-t} [ t+1 ; t ]

What is this phase portrait like? There are the eigensolutions.
The solution  u_2  has  u_2(0) = w = [ 0 ; 1 ] .  When  t  increases
away from zero, the  x-coordinate becomes positive. All solutions decay
to zero as  t  grows. You can see that as  t increases the trajectory of
u_2(t)  becomes tangent to the eigenline. It hooks around and comes
in towards zero from the other direction. This lets you fill out this
phase portrait, which is called a "defective nodal sink."

This isn't the only thing that can happen if you have a repeated eigenvalue,
though. What if  A = [ 2 0 ; 0 2 ] , for example? This is upper triangular,
so the eigenvalues are the diagonal entries, 2 and 2 : repeated. To find
an eigenvector we subtract  2  from the diagonal entries - you get the zero
matrix. So every vector is an eigenvector! This is the "complete case."
Every ray from the origin is the trajectory of a solution. The phase portrait
is called a "star source."

To summarize: if there is a repeated eigenvalue (in the 2 x 2 case)
then either:
the matrix is diagonal, when the matrix is "complete" an the phase portrait
is a star,    or
the matrix is not diagonal, when the matrix is "defective," you have to
use the algorithm described above to find a second independent solution,
and the phase portrait is a defective node.

Another marginal case is:

det = 0 :  at least one of the eigenvalues is zero.
If  v  is an eigenvector corresponding to this eigenvalue, then
the constant vector valued function  u(t) = c alpha  is a solution for
any constant  c:  there is a line (at least) of constant solutions.
Several patterns are possible, and they are illustrated in the
Supplementary Notes.

The phase portrait in each one of these borderline cases shows some
features which are not determined purely by the eigenvalues.
Another feature for which you need to go back to the matrix is deciding
on whether a spiral is turning clockwise or counterclockwise.
(This is independent of the question of whether it is a sink or a source.)

We can tell which you get by thinking about what  u' is at some point.
A convenient one to pick is  [1;0] :  then  u' = Au = (first column of  A).
So if the bottom left entry is positive, the spiral is moving
counterclockwise; if negative, it is moving clockwise.

[4] (I did not talk about:)

Stability: All linear systems fall into one of the following categories:

Asymptotically stable: all solutions ---> 0 as t ---> infinity
These systems occupy the upper left quadrant, tr < 0 and det > 0, so the eigenvalues have negative real part.

Unstable: most solutions ---> infinity as t ---> infinity
Saddles and unstable nodes and spirals are examples.

Neutrally stable: all solutions stay bounded as t ---> infinity but most do not ---> 0 .
These systems occur along the boundary between the other two types:
-- Along the ray tr = 0 , det > 0, so the eigenvalues are nonzero and purely imaginary. The phase portrait is a "center" and the trajectories are ellipses.
-- Along the ray det = 0 , tr < 0 , so one eigenvalue is zero and the other is negative. The phase portrait is a comb.