18.034 FINAL EXAM
MAY 20, 2004

Name: ____

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Total:	________	/200

Instructions: *Please write your name at the top of every page of the exam.* The exam is closed book, closed notes, and calculators are not allowed. You will have approximately 3 hours for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown.

You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., raise your hand.

Date: Spring 2004.

Table of Laplace Transforms

	$y(t)$	$Y(s) = \mathcal{L}[y(t)]$
1.	$y^{(n)}(t)$	$s^nY(s) - (y^{(n-1)}(0) + \cdots + s^{n-1}y(0))$
2.	t^n	$n!/s^{n+1}$
3.	$t^n y(t)$	$(-1)^n Y^{(n)}(s)$
4.	$\cos(\omega t)$	$s/(s^2 + \omega^2)$
5.	$\sin(\omega t)$	$\omega/(s^2 + \omega^2)$
6.	$e^{at}y(t)$	$Y(s-a)$
7.	$y(at),\ a > 0$	$\frac{1}{a}Y(s/a)$
8.	$S(t-t_0)y(t-t_0),\ t_0 \geq 0$	$e^{-st_0}Y(s)$
9.	$\delta(t-t_0),\ t_0 \geq 0$	e^{-st_0}
10.	$(S(t)y) * (S(t)z)$	$Y(s)Z(s)$
11.	$y(t),\ y(t+T) = y(t)$	$\frac{1}{1-e^{-sT}} \int_0^T e^{-st}y(t)dt$

Name: ________________________ **Problem 1**: ____________ /10

Problem 1(10 points) Two objects of mass m are connected to a rigid base and to each other as shown on the previous page. The spring connecting each object to the base has constant k, and the spring connecting the objects to each other has constant $2k$. Denote by x_1 the displacement of the object on the left from equilibrium (displacement to the right = positive displacement). Denote by x_2 the displacement of the object on the right from equilibrium (displacement to the right = positive displacement). Denote $\omega = \sqrt{k/m}$.

(a)(5 points) Find a system of 2^{nd} order linear ODEs satisfied by x_1 and x_2 of the form,

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

In other words, find the matrix A.

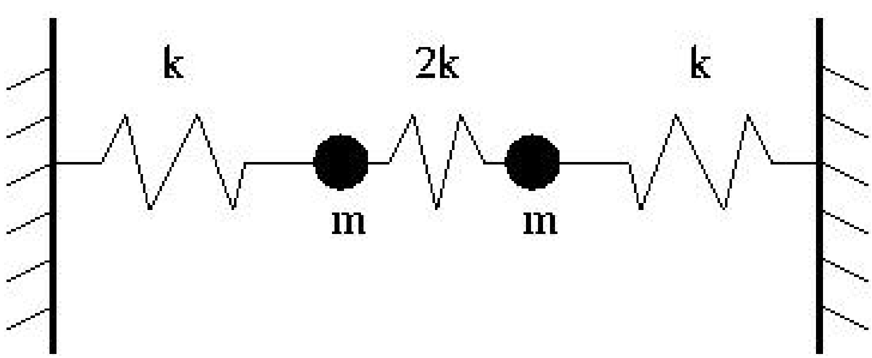
Name: ______________________________ **Problem 1, contd.**

(b)(5 points) Introduce new variables $v_1 = x_1'$ and $v_2 = x_2'$. Find a system of 1^{st} order linear ODEs satisfied by x_1, v_1, x_2 and v_2 of the form,

$$\begin{bmatrix} x_1' \\ v_1' \\ x_2' \\ v_2' \end{bmatrix} = B \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}.$$

In other words, find the matrix B.

Extra credit(2 points) What is the relationship of $p_A(\lambda)$ and $p_B(\lambda)$?



Name: ______________________ **Problem 2**: __________ /20

Problem 2(20 points) Consider the ODE,

$$y(t)' + \frac{2}{t}y(t) = 3e^{-t^3/3}, \quad t > 0$$

(a)(5 points) Find an integrating factor.

(b)(10 points) Find the general solution.

(c)(5 points) Find the unique solution that has an extension to a continuous function on $[0, \infty)$.

Name: ______________________________ **Problem 3**: ____________ /25

Problem 3(25 points) A basic solution pair of the homogeneous linear 2^{nd} order ODE,

$$y'' + \frac{2t}{t^2-4}y' - 16\frac{1}{(t^2-4)^2}y = 0, \quad t > 2$$

is given by $\{y_1(t), y_2(t)\}$,

$$y_1(t) = \frac{t-2}{t+2}, \quad y_2(t) = \frac{t+2}{t-2}.$$

(a)(10 points) Compute the Wronskian $W[y_1, y_2](t)$.

Name: ________________________________ **Problem 3, contd.**

(b)(15 points) Use variation of parameters to find a particular solution of the inhomogeneous ODE,

$$y'' + \frac{2t}{t^2-4}y' - 16\frac{1}{(t^2-4)^2}y = 1.$$

Name: ______________________________________ **Problem 4**: ________________ /15

Problem 4(15 points) Using the method of undetermined coefficients and the exponential shift rule, find a particular solution of the inhomogeneous linear 2^{nd} order ODE,

$$y'' + 5y' + 6y = -4te^{-3t}.$$

Name: ______________________________ **Problem 5**: ____________ /20

Problem 5(20 points) On the interval $[0, 2)$, let $f(t) = t + 1$. Denote by $\widetilde{f}(t)$ the even extension of $f(t)$ as a periodic function of period 4. Denote by $\mathrm{FCS}[\widetilde{f}]$ the Fourier cosine series of $\widetilde{f}(t)$.

(a)(5 points) Graph $\mathrm{FCS}[\widetilde{f}]$ on the interval $[-4, 4]$. Make special note of all discontinuities and the *actual value* of $\mathrm{FCS}[\widetilde{f}]$ at these points.

Name: ______________________________ **Problem 5, contd.**

(b)(10 points) An orthonormal basis for the even periodic functions of period 4 is,

$$\phi_0(t) = \frac{1}{2}, \quad \phi_n(t) = \frac{1}{\sqrt{2}} \cos(n\pi t/2), \ n = 1, 2, 3, \ldots$$

Compute the Fourier coefficients,

$$a_n = \langle \widetilde{f}, \phi_n \rangle = \int_{-2}^{2} \widetilde{f}(t)\phi_n(t)dt,$$

and express your answer as a Fourier cosine series,

$$\widetilde{f}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{2}} \cos(n\pi t/2).$$

Don't forget to compute a_0.

Extra credit(3 points) Plug in $t = 0$ to get a formula for the series,

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}.$$

Name: ________________________ **Problem 6**: ____________ /25

Problem 6(25 points) Let $f(t)$ be the piecewise continuous function,

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ e^{-3(t-1)}, & t \geq 1 \end{cases}$$

Let $y(t)$ be the continuously differentiable and piecewise twice-differentiable solution of the following IVP,

$$\begin{cases} y'' + 5y' + 6y = f(t), \\ y(0) = 0, \\ y'(0) = 0 \end{cases}$$

Denote by $Y(s)$ the Laplace transform,

$$\mathcal{L}[y(t)] = \int_0^\infty e^{-st} y(t) dt.$$

(a)(5 points) Compute the Laplace transform of the IVP and use this to find an equation that $Y(s)$ satisfies.

(b)(10 points) Solve the equation fo $Y(s)$ and find the partial fraction decomposition of your answer. Use the Heaviside cover-up method to simplify the partial fraction decomposition.

Name: ________________________________ **Problem 6, contd.**

(c)(10 points) Find $y(t)$ by computing the inverse Laplace transform.

Question:(Not to be answered) Is there a simpler solution than using the Laplace transform? If so, you can use this to double-check your answer.

Name: ______________________ **Problem 7**: __________ /10

Problem 7 Let A be the real 3×3 matrix,

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a)(3 points) Compute the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$.

(b)(7 points) For each eigenvalue, find an eigenvector (*not a generalizaed eigenvector*).

Name: ______________________________________ **Problem 7, contd.**

Extra credit(3 points) For one of the eigenvalues, the eigenspace is deficient. Find a generalized eigenvector that is not an eigenvector.

Name: ________________________ **Problem 8**: __________ /35

Problem 8(35 points) The linear system $\mathbf{x}' = A\mathbf{x}$ is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a)(5 points) Compute Trace(A), det(A), the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$ and the eigenvalues of A.

(b)(10 points) For each eigenvalue (or for one eigenvalue from each complex conjugate pair), find an eigenvector.

Name: ______________________________ **Problem 8, contd.**

(c)(5 points) Find the general solution of the linear system of ODEs. Write your answer in the form of a solution matrix $X(t)$ whose column vectors are a basis for the solution space.

(d)(5 points) Compute the exponential matrix,

$$\exp(tA) = X(t)X(0)^{-1}.$$

Name: ________________________________ **Problem 8, contd.**

(e)(10 points) Denote by $\mathbf{f}(t)$ the vector-valued function,

$$\mathbf{f}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}.$$

Denote by $\mathbf{x}_0$ the column vector,

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For the following IVP write down the solution in terms of the matrix exponential.

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \\ \mathbf{x}(0) = \mathbf{x}_0. \end{cases}$$

Compute the entries of the constant term vector and the integrand column vector, but do not evaluate the integrals.

Name: ________________________________ **Problem 9**: ______________ /40

Problem 9(40 points) Consider the following nonlinear, autonomous planar system,

$$\begin{cases} x' &= 12x(y-1) \\ y' &= 2y(x+y-2) \end{cases}$$

(a)(5 points) Find all equilibrium points.

(b)(5 points) Determine the linearization at each equilibrium point.

Name: ______________________________ **Problem 9, contd.**

(c)(15 points) For each linearization, determine the eigenvalues. If the eigenvalues are complex conjugates, determine the rotation (clockwise in/out, counterclockwise in/out). If the eigenvalues are real, determine roughly the eigenvectors and the type of the local phase portrait.

Name: ______________________________________ **Problem 9, contd.**

For the following 2 parts, **please sketch your answer on the graph on the following page**.

(d)(5 points) Using a dashed line, sketch the x-nullcline and y-nullcline. Draw a few representative arrows indicating the direction of the orbits on the nullcline on each side of each equilibrium point.

(e)(10 points) Sketch the phase portrait. Label all equilibrium points. For each equilibrium point, sketch a few orbits. In particular, for each saddle sketch each orbit whose limit or inverse limit is the equilibrium point.

There is one basin of attraction. Use bold lines to indicate each (rough) separatrix bounding this basin of attraction. Your sketch should just be a rough sketch, but it should be qualitatively correct.

Name: ______________________________ **Problem 10**: ____________ /10

Problem 10, Extra Credit Problem(10 points) Let $a(t)$ and $b(t)$ be continous functions on an interval (c, d). Let $\{y_1, y_2\}$ be a basic solution pair on this interval of the ODE,

$$y'' + a(t)y' + b(t)y = 0.$$

Prove that between every two zeroes of y_1 there is a zero of y_2 (and vice versa).