18.034 PRACTICE FINAL EXAM, SPRING 2004

The final exam will be held on Thursday, May 20, 9:00AM–12:00NOON. The final exam will be closed notes, closed book, and calculators will not be permitted. Scratch paper and a stapler will be available. A short list of Laplace transforms will be provided (the same as the list on Exam 3). The following problems are representative of the problems on the exam.

Problem 1 Two different chemical solutions are pumped into a container of volume $100\ L$, each at the rate of $5\ L/s$, and thoroughly mixed solution is pumped out at the rate of $10\ L/s$. The inflow concentration of Chemical 1 is q_1 and the inflow concentration of Chemical 2 is q_2. Denote the mass of Chemical 1 by x_1 and the mass of Chemical 2 by x_2. A catalyst in the container transforms Chemical 1 into Chemical 2 at a rate $0.4x_1$ per second.

(a) Determine the 2×2 linear system of 1st order ODEs that x_1 and x_2 satisfy.

(b) Without finding the general solution, determine the steady-state solution for x_1 and x_2.

Problem 2 Consider the following inhomogeneous linear 1st order ODE on the interval $t > 1$.

$$y' + \frac{1}{t^2 - 1}y = \sqrt{\frac{t+1}{t-1}}.$$

(a) Let $a > 0$ be a real number, and let $u(t)$ be the function,

$$u(t) = \left(\frac{t-1}{t+1}\right)^a.$$

Compute $u'/u = \frac{d}{dt}\ln(u(t))$.

(b) Find a value of a such that $u(t)$ is an integrating factor.

(c) Find the general solution of the ODE.

(d) Find the unique solution $y(t)$ such that the limit,

$$\lim_{t \to 1^+} y(t),$$

exists and is bounded.

Problem 3 A basic solution pair of the homogeneous linear 2nd order ODE,

$$y'' + \frac{2t}{t^2 - 1}y' - 16\frac{1}{(t^2 - 1)^2}y = 0,$$

is given by $\{y_1, y_2\}$,

$$y_1(t) = \left(\frac{t-1}{t+1}\right)^2, \quad y_2(t) = \left(\frac{t+1}{t-1}\right)^2.$$

(a) Compute the Wronskian $W[y_1, y_2](t)$.

(b) Use variation of parameters to find a particular solution of the inhomogeneous ODE,

$$y'' + \frac{2t}{t^2 - 1}y' - 16\frac{1}{(t^2 - 1)^2}y = t^2 - 1.$$

Problem 4 Using the method of undetermined coefficients, find a particular solution of the inhomogeneous linear 2nd order ODE,

$$y'' + 4y' + 5y = 5e^{-2t}\cos(t).$$

Problem 5 On the interval $[0, \pi)$, let $f(t) = \cos(2t)$. Denote by $\widetilde{f}(t)$ the odd extension of $f(t)$ as a periodic function of period 2π. Denote by $\text{FSS}[\widetilde{f}]$ the Fourier sine series of $\widetilde{f}(t)$.

(a) Graph $\text{FSS}[\widetilde{f}]$ on the interval $[-3\pi, 3\pi]$. Make special note of all discontinuities and the *actual value* of $\text{FSS}[\widetilde{f}]$ at these points (which does not necessarily agree with the value of $\widetilde{f}$).

(b) An orthonormal basis for the odd periodic functions of period 2π is,

$$\phi_n(t) = \frac{1}{\sqrt{\pi}}\sin(nt), \ \ n = 1, 2, 3, \ldots$$

Compute the Fourier coefficients,

$$a_n = \langle \widetilde{f}, \phi_n \rangle = \int_{-\pi}^{\pi} \widetilde{f}(t)\phi_n(t)dt,$$

and express your answer as a Fourier sine series,

$$\widetilde{f}(t) = \sum_{n=1}^{\infty} a_n\phi_n(t) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\pi}}\sin(nt).$$

Play close attention to $n = 2$.

Problem 6 On the interval $[-\pi, \pi)$, let $f(t)$ be the square-wave function,

$$f(t) = \begin{cases} 1, & 0 \le t < \pi, \\ -1, & -\pi \le t < 0 \end{cases}$$

Let $\widetilde{f}(t)$ be the extension of $f(t)$ to a periodic function of period 2π. An orthonormal basis for the periodic functions of period 2π is,

$$\phi_n(t) = \frac{1}{\sqrt{2\pi}}e^{int}, \ \ n \in \mathbb{Z}.$$

(a) Compute the Fourier coefficients,

$$a_n(\widetilde{f}) = \langle \widetilde{f}, \phi_n \rangle = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}}f(t)e^{-int}dt.$$

(b) Denote by $y(t)$ the periodic function of period 2π that solves the ODE,

$$y'' + 4y' + 5y = \widetilde{f}(t).$$

Using (a), compute the Fourier coefficients,

$$a_n(y) = \langle y, \phi_n \rangle.$$

(c) Write down just the terms in the Fourier series of $y(t)$ coming from $n = -1$ and $n = 1$. Express in terms of sine and cosine.

Problem 7 Consider the IVP,

$$\begin{cases} y'' - 4y' + 5y = 3e^{2t}\sin(t), \\ y(0) = 1, \\ y'(0) = 0 \end{cases}$$

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Denote by $Y(s)$ the Laplace transform,

$$\mathcal{L}[y(t)] = \int_0^\infty e^{-st} y(t) dt.$$

(a) Compute the Laplace transform of the IVP and use this to find an equation that $Y(s)$ satisfies.

(b) Solve the equation for $Y(s)$. No partial fractions are needed.

(c) Find $y(t)$ by computing the inverse Laplace transform. Rewrite the following terms

$$\frac{1}{((s-a)^2 + \omega^2)^2} = \frac{1}{(s-a)^2 + \omega^2} \times \frac{1}{(s-a)^2 + \omega^2}.$$

Therefore the inverse Laplace transform is,

$$\mathcal{L}^{-1}\left[\frac{1}{((s-a)^2 + \omega^2)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s-a)^2 + \omega^2}\right] * \mathcal{L}^{-1}\left[\frac{1}{(s-a)^2 + \omega^2}\right].$$

Do not evaluate the integral for this convolution – leave it as a convolution.

Problem 8 The linear system $\mathbf{x}' = A\mathbf{x}$ is,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) Compute Trace(A), $\det(A)$, the characteristic polynomial $p_A(\lambda) = \det(\lambda I - A)$ and the eigenvalues of A.

(b) For each eigenvalue (or for one eigenvalue from each complex conjugate pair), find an eigenvector.

(c) Find the general solution of the linear system of ODEs. Write your answer in the form of a solution matrix $X(t)$ whose column vectors are a basis for the solution space.

(d) Compute the exponential matrix,

$$\exp(tA) = X(t)X(0)^{-1}.$$

(e) Denote by $\mathbf{f}(t)$ the vector-valued function,

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ 3e^{2t}\sin(t) \end{bmatrix}.$$

Denote by $\mathbf{x}_0$ the column vector,

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For the following IVP write down the solution in terms of the matrix exponential.

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

Compute the entries of the constant term vector and the integrand column vector, but do not evaluate the integrals.

Problem 9 Consider the following nonlinear, autonomous planar system,

$$\begin{cases} x' &= (x+y)(x-1) \\ y' &= (x-y)(x+1) \end{cases}$$

(a) Find all equilibrium points.

(b) Determine the linearization at each equilibrium point.

(c) For each linearization, determine the eigenvalues. If the eigenvalues are complex conjugates, determine the rotation (clockwise in/out, counterclockwise in/out). If the eigenvalues are real, determine roughly the eigenvectors and the type of the local phase portrait.

(d) Using a dashed line, sketch the x-nullcline and y-nullcline. Draw a few representative arrows indicating the direction of the orbits on the nullcline on each side of each equilibrium point.

(e) Sketch the phase portrait. Use bold lines to indicate each (rough) separatrix. Label the basins of attraction (with the numbers 1, 2, 3, etc. placed at some point in the basin of attraction). Your sketch should just be a rough sketch, but it should be qualitatively correct.