18.034 Honors Differential Equations
Spring 2009

Problem set 6, Solution keys

1. (a) $\int_0^\infty e^{-t}t^r dt = \int_0^1 e^{-t}t^r dt + \int_1^\infty e^{-t}t^r dt := (I) + (II).$

   For $r > -1$.

   $(I) \le \int_0^1 t^r dt < +\infty.$

   $(II) \simeq \sum_{n=1}^\infty e^{-n}n^r < +\infty$

   (b)

$$\begin{aligned}\Gamma(r+1) = \int_0^\infty e^{-t}t^r dt &= -e^{-t}t^r|_o^\infty + \int_0^\infty re^t t^{r-1}dt \quad \text{(Integration by parts)} \\ &= r\Gamma(r)\end{aligned}$$

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1, \qquad \Gamma(1/2) = \int_0^\infty e^{-t}t^{-1/2}dt = \int_0^\infty 2e^{-u^2}du = \sqrt{\pi}$$

   (c) $\mathcal{L}(t^r) = \int_0^\infty e^{-t}t^r dt = \int_0^\infty e^{-u}(u/s)^r(1/s)du = (\frac{1}{s^{r+1}})\int_0^\infty e^{-u}(u)^r du = \frac{\mathcal{L}(r+1)}{s^{r+1}}$

2. (This is a long one.)

   (a)

$$\begin{aligned}\mathcal{L}[h(t-c)\sin t] &= \int_0^\infty e^{-st}\sin t dt \\ &= e^{-sc}\mathcal{L}[\sin(t+c)] \\ &= e^{-sc}\mathcal{L}[\sin t\cos c + \cos t \sin c] \\ &= e^{-sc}\frac{\cos c + s\sin c}{s^2+1}\end{aligned}$$

   Taking the transform,

$$\begin{aligned}\mathcal{L}(y) &= \frac{1}{s^2+w^2}(\frac{1}{s^2+1} - e^{-sc}\frac{\cos c + s\sin c}{s^2+1}) \\ &= \frac{1}{w^2-1}(\frac{1}{s^2+1} - \frac{1}{s^2+w^2})(1-e^{-sc}(\cos c + s\sin c))\end{aligned}$$

   So, $y(t) = \frac{1}{w^2-1}(\sin t - \frac{\sin wt}{w} - h(t-c)(\sin t - \frac{\sin w(t-c)}{w}\cos c - \cos w(t-c)\sin c))$

   (b) $y(c) = \frac{1}{w^2-1}(\sin c - \frac{\sin wc}{w}), \qquad y'(c) = \frac{1}{w^2-1}(\cos c - \cos wc).$

   (c) $y''(c+) - y''(c-) = \frac{-1}{w^2-1}(-\sin c + w^2\sin c).$

3. Let $f_0(t) = \sin t \qquad 0 \le t < \pi.$ $\qquad\qquad \mathcal{L}f_0 = \frac{1}{s^2+1}(1+e^{-\pi s})$

   $\mathcal{L}f = \frac{\mathcal{L}f_0}{1-e^{-\pi s}} = \frac{1}{s^2+1}\frac{1+e^{-\pi s}}{1-e^{-\pi s}}.$

   Alternatively, compute term by term in $f(t) = \sin t + 2\sum_{n=1}^{\infty} h(t-n\pi)\sin(t-n\pi)$

4. (a) Taking the transform, $s^2\mathcal{L}y - (sa+b) + \mathcal{L}y = Ae^{-sc.}$
   So, $y(t) = Ah(t-c)\sin(t-c) + a\cos t + b\sin t.$

   (b) The reason why $y(c) = 0$ is necessary is: at a point $y(c) = 0$, the impulse can control the derivative and reduce $y'(c)$ to be zero. Then $y(t) = 0$ for $t > c$ by uniqueness. But, at a point $y(c) \ne 0$, no choice of the impulse will make $y'(c) = 0$.
   In general in $y'' + ay' + by = f(t)$, a, b constants, the effect of $f(t) \to f(t) + \delta(t)$ is the same as the effect of $y'(0) \to y'(0) + c$.

5. (a) Uses $\displaystyle \frac{d}{dt}\int_{t_0}^{t} f(s,t)ds = f(t,t) + \int_{t_0}^{t} \frac{\delta f}{\delta t}(s,t)dt.$

   (b) Taking the transform, $\mathcal{L}y + \frac{1}{s^2}\mathcal{L}y = -\frac{1}{2}\frac{1}{s^2+4}$, and $\mathcal{L}y = -\frac{1}{2}\frac{s^2}{(s^2+1)(s^2+4)}.$
   So, $y(t) = \frac{1}{6}\sin t + \frac{1}{3}\sin 2t$

6. (a) Uses $\mathcal{L}[y''] = -\frac{d}{ds}(s^2Y(s)) = -s^2Y'(s) - 2sY(s).$

   (b) (This is again long.)

$$\frac{Y'}{Y} = -\frac{s}{s^2+1}, \quad \text{so} \quad Y(s) = c(1+s^2)^{-\frac{1}{2}}$$

   Using the binomial series for $(1+s^2)^{-\frac{1}{2}}$, $Y(s) = \frac{c}{s}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}s^{-2n} = c\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\frac{(2n)!}{s^{2n+1}}.$

   Since $\mathcal{L}[t^{2n}] = \dfrac{(2n)!}{s^{2n+1}}$, $y(t) = c\sum_{n=0}^{\infty}\dfrac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$

   Clearly, $y(0) = 0$. $y^{(k)}(0) = \begin{cases} 0 & \text{k=2n+1} \\ \frac{(-1)^n(2n)!}{2^{2n}(n!)^2} & \text{k=2n} \end{cases}$