18.03SC Practice Problems 6

Complex numbers

Solution Suggestions

1. *Mark* $z = 1 + \sqrt{3}i$ *on the complex plane. What are its polar coordinates? Then mark* z^n *for* $n = 1, 2, 3, 4$*. What is each in the form* $a + bi$*? What is each one in the form* $Ae^{i\theta}$*? Then mark* z^n *for* $n = 0, -1, -2, -3, -4$*.*

Here is a picture of z marked on the complex plane.

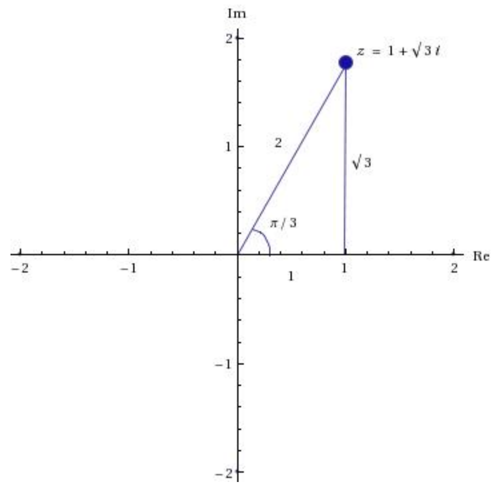

Figure 1: $z = 1 + \sqrt{3}i$ on the complex plane.

The complex number z has radius (a.k.a. modulus or magnitude) $r = |z| = \sqrt{1+3} = 2$ and, from the figure, angle (a.k.a. argument) $\theta = \text{Arg}(z) = 60° = \pi/3$ radians. That is, z has polar coordinates $(r, \theta) = (2, \pi/3)$.

Recall that that when multiplying complex numbers, "magnitudes multiply, arguments add." Compute the magnitude and argument of z^n for $n = 1, 2, 3, 4$ as in the following table.

n	$\lvert z^n \rvert = \lvert z \rvert^n$	$\text{Arg}(z^n) = n \cdot \text{Arg}(z)$
1	2	$\pi/3$
2	4	$2\pi/3$
3	8	π
4	16	$4\pi/3$

Use this table to mark these positive powers of z on the complex plane as in Figure 2. Note that the second figure has a different scale than the first one.

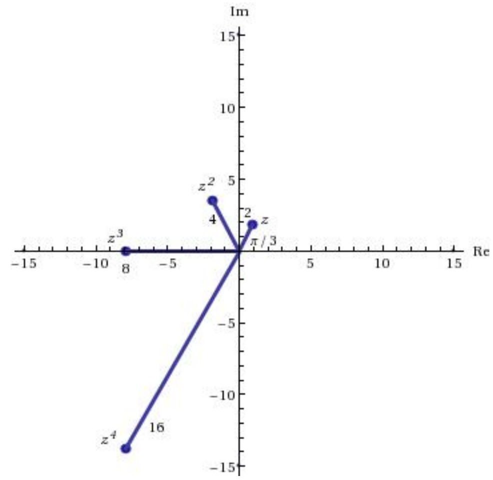

Figure 2: z^n for $n = 1, 2, 3, 4$ on the complex plane.

In rectangular form, $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$. Thus

$$z^2 = 4(\cos(2\pi/3) + i\sin(2\pi/3)) = -2 + 2\sqrt{3}i,$$

$$z^3 = 8(-1) = -8,$$

$$z^4 = 16(\cos(4\pi/3) + i\sin(4\pi/3)) = -8 - 8\sqrt{3}i,$$

which match the figure.

In polar form, $z^n = r^n e^{in\theta}$. So $z = 2e^{i\pi/3}$, $z^2 = 4e^{i2\pi/3}$, $z^3 = 8e^{i\pi}$, $z^4 = 16e^{i4\pi/3}$, which we could have read off from the table.

Now, $z^0 = 1$, and negative powers have inverse radius and negative argument of the positive powers: $z^{-n} = r^{-n}e^{-in\theta}$ is on the radial line of $-n\pi/3$ with radius 2^{-k} for $k = 1, 2, 3, 4$. Use this to mark each on the complex plane as in Figure 3.

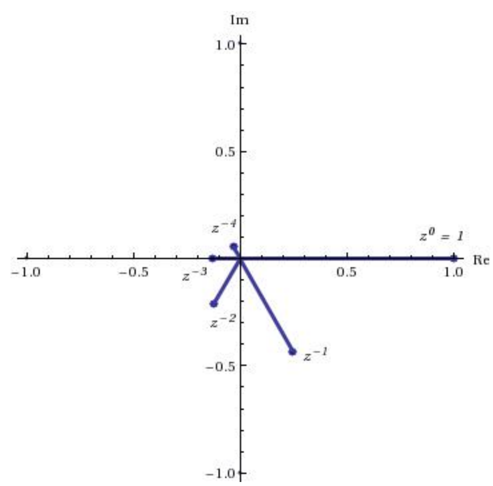

Figure 3: z^n for $n = 0, -1, -2, -3, -4$ on the complex plane.

2. *Find a complex number $a + bi$ such that $e^{a+bi} = 1 + \sqrt{3}i$. In fact, find all such complex numbers. For definiteness, fix b to be positive but as small as possible. (This is probably the* *first one you thought of.) What is* $e^{n(a+bi)}$ *for* $n = 1, 2, 3, 4$*? (Hint:* $e^{n(a+bi)} = (e^{a+bi})^n$*.) How about for* $n = 0, -1, -2, -3, -4$*?*

The complex number $e^{a+bi} = e^a e^{bi}$ has modulus e^a and argument b. The modulus of a complex number is uniquely defined, while the argument is only determined up to adding multiples of 2π. So if $e^{a+bi} = 1 + \sqrt{3}i$, we must have $e^a = 2$ and $b = \pi/3 + 2k\pi$ for any integer k. Thus, $a + bi$ can be any complex number of the form $\ln 2 + i(\pi/3 + 2k\pi)$ for some integer k. The smallest positive value of b is $\pi/3$, so take

$$a + bi = \ln 2 + i\pi/3.$$

Following the hint, $e^{n(a+bi)} = (1 + \sqrt{3}i)^n$, which we computed in Question 1. That is, for $n = 1, 2, 3, 4$, this is $1 + \sqrt{3}i, \quad -2 + 2\sqrt{3}i, \quad -8, \quad -8 - 8\sqrt{3}i$, respectively, and for $n = 0, -1, -2, -3, -4$, we have 1, $2^{-1}e^{-i\pi/3} = \frac{1-\sqrt{3}i}{4}$, $2^{-2}e^{-i2\pi/3} = \frac{-1-\sqrt{3}i}{8}$, $2^{-3}e^{-i\pi} = -1/8$, and $2^{-4}e^{-i4\pi/3} = \frac{-1+\sqrt{3}i}{32}$. Note that we did not actually need the values we found for a and b to answer this part of the question.