Review of Vectors and Matrices

1. Vectors

A **vector** (or n-vector) is an n-tuple of numbers; they are usually real numbers, but we will sometimes allow them to be complex numbers. All the rules and operations below apply just as well to n-tuples of complex numbers. (In the context of vectors, a single real or complex number, i.e., a constant, is called a **scalar**.) As we are dealing with 2×2 linear systems, we are primarily interested in scalars and 2-vectors: ordered pairs of numbers.

The pair can be written horizontally as a **row vector** or vertically as a **column vector**. In these notes, it will almost always be a column. To save space, we will sometimes write the column vector as shown below; the small T stands for **transpose**, and means: change the row to a column.

$$\mathbf{a} = (a, b) \quad \textit{row vector} \qquad\qquad \mathbf{a} = (a, b)^T \quad \textit{column vector}$$

These notes use boldface for vectors; in handwriting, place an arrow $\vec{a}$ over the letter.

Vector operations. Here are two standard operations on vectors:

- addition: $(a, b) + (c, d) = (a + c, b + d)$.
- multiplication by a scalar: $c\,(a, b) = (ca, cb)$
- scalar product: $(a, b)(c, d) = ac + bd$

2. Matrices

An $\mathbf{m \times n}$ **matrix** A is a rectangular array of numbers (real or complex) having m rows and n columns. The element in the i-th row and j-th column is called the ij**-th entry** and written a_{ij}. The matrix itself is sometimes written (a_{ij}), i.e., by giving its generic entry, inside the matrix parentheses. We will be interested in matrices where m and n are at most 2.

Note that a 1×2 matrix is a row vector; an 2×1 matrix is a column vector.

Matrix operations.

- addition: if A and B are both $m \times n$ matrices, they are added by adding the corresponding entries; i.e., if $A = (a_{ij})$ and $B = (b_{ij})$, then $A + B = (a_{ij} + b_{ij})$.

- multiplication by a scalar: to get cA, multiply every entry of A by the scalar c; i.e., if $A = (a_{ij})$, then $cA = (ca_{ij})$.
- matrix multiplication: if A is an $m \times n$ matrix and B is an $n \times k$ matrix, their product AB is an $m \times k$ matrix, defined by using the scalar product operation:

  $$ij\text{-th entry of } AB = (i\text{-th row of } A)(j\text{-th column of } B)^T$$

  where the scalar product of two 1-vectors is just their normal product.

The definition makes sense since both vectors on the right are vectors of the same length n. In what follows, the most important cases of matrix multiplication will be:
(i) A and B are square 2×2 matrices. In this case, multiplication is always possible, and the product AB is again an 2×2 matrix.
(ii) A is an 2×2 matrix and $B = \mathbf{b}$, a column 2-vector. In this case, the matrix product $A\mathbf{b}$ is again a column 2-vector.

Laws satisfied by the matrix operations.
For any matrices for which the products and sums below are defined, we have

$(A\,B)\,C$	$= A\,(B\,C)$		*(associative law)*
$A\,(B+C)$	$= A\,B + A\,C,$	$(A+B)\,C = A\,C + A\,C$	*(distributive laws)*
$A\,B$	$\neq B\,A$		*(commutative law fails in general)*

The **identity matrix** I is the 2×2 matrix with 1's on the main diagonal (upper left and bottom right), and 0's elsewhere. If A is an arbitrary 2×2 matrix, it is easy to check from the definition of matrix multiplication that

$$AI = A \quad \text{and} \quad IA = A.$$

The exercises later in this session should help you get familiar with all these concepts.