18.04 Practice problems exam 2, Spring 2018 Solutions

Problem 1. Harmonic functions

(a) *Show $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2$ is harmonic and find a harmonic conjugate.*

It's easy to compute:

$$u_x = 3x^2 - 3y^2 + 6x, \qquad u_{xx} = 6x + 6$$
$$u_y = -6xy - 6y, \qquad u_{yy} = -6x - 6$$

It's clear that $\nabla^2 u = u_{xx} + u_{yy} = 0$, so u is harmonic.

If v is a conjugate harmonic function to u, then $u + iv$ is analytic and the Cauchy-Riemann equations tell us that $v_x = -u_y$ and $v_y = u_x$. Therefore, we can integrate u_x and u_y to find v.

$$v_x = -u_y = 6xy + 6y \qquad \Rightarrow\ v = 3x^2 y + 6xy + g(y)$$
$$v_y = u_x = 3x^2 - 3y^2 + 6x \qquad \Rightarrow\ v = 3x^2 y - y^3 + 6xy + h(x)$$

Comparing the two expressions for v we see that $g(y) = -y^3 + C$ and $h(x) = C$. So $\boxed{v = 3x^2 y - y^3 + 6xy + C}$.

(b) *Find all harmonic functions u on the unit disk such that $u(1/2) = 2$ and $u(z) \geq 2$ for all z in the disk.*

Solution: The only possibility is the constant function $u(z) \equiv 2$. The maximum principle for harmonic functions says that if u takes a relative maximum or minimum at an interior point then it is constant. (This is a consequence of the mean value theorem.)

(c) *The temperature of the boundary of the unit disk is maintained at $T = 1$ in the first quadrant, $T = 2$ in the second quadrant, $T = 3$ in the third quadrant and $T = 4$ in the fourth quadrant. What is the temperature at the center of the disk*

Solution: The mean value theorem says that $f(0)$ is the average over any circle centered at 0. This is clearly the average of the (constant) values in each quadrant. So $f(0) = 2.5$.

(d) *Show that if u and v are conjugate harmonic functions then uv is harmonic.*

Solution: **Easy method.** We know $f = u + iv$ is analytic. Therefore $f^2 = u^2 - y^2 + 2iuv$ is also analytic. So, $\mathrm{Im}(f^2) = 2uv$ is harmonic. QED

Calculation method.

$$(uv)_{xx} = u_{xx}v + 2u_x v_x + uv_{xx}$$
$$(uv)_{yy} = u_{yy}v + 2u_y v_y + uv_{yy}$$

We know u, v are harmonic and satisfy the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$. So adding the above equations we get

$$(uv)_{xx} + (uv)_{yy} = (u_{xx} + u_{yy})v + 2(-u_x u_y + u_x u_y) + u(v_{xx} + v_{yy}) = 0.$$

We have shown that uv is harmonic.

(e) *Show that if u is harmonic then u_x is harmonic.*

Solution: **Easy method.** For some conjugate v, $f = u + iv$ is harmonic. Since $f' = u_x + iv_x$, we know $\text{Re}(f') = u_x$ is harmonic.

Direct calculation $(u_x)_{xx} + (u_x)_{yy} = (u_{xx} + u_{yy})_x = 0$.

(f) *Show that if u is harmonic and u^2 is harmonic the u is constant.*

(We always assume harmonic functions are real valued.)

Solution: We calculate this directly.

$$(u^2)_{xx} = 2(u_x)^2 + 2uu_{xx}, \qquad (u^2)_{yy} = 2(u_y)^2 + 2uu_{yy}.$$

Assume that u and u^2 are harmonic, then

$$0 = (u^2)_{xx} + (u^2)_{yy} = 2((u_x)^2 + (u_y)^2) + 2u(u_{xx} + u_{yy}) = 2((u_x)^2 + (u_y)^2).$$

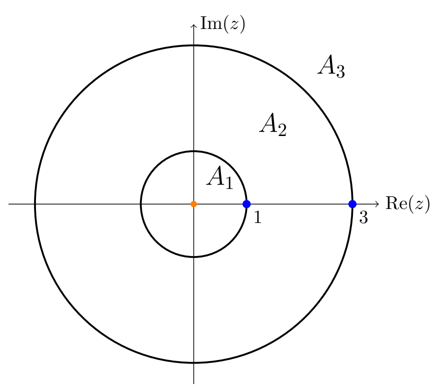
As a sum of squares, $(u_x)^2 + (u_y)^2 = 0$ implies $u_x = u_y = 0$. This implies u is constant. QED.

Problem 2.

Let $f(z) = \dfrac{1}{(z-1)(z-3)}$. Find Laurent series for f on each of the 3 annular regions centered at $z = 0$ where f is analytic.

Solution: The poles are at $z = 1$ and $z = 3$. This divides the plane into 3 annular regions, with f analytic on each region:

$$A_1 : |z| < 1, \qquad A_2 : 1 < |z| < 3, \qquad A_3 : 3 < |z|.$$



Using partial fractions we get $f(z) = -\frac{1}{2} \cdot \frac{1}{z-1} + \frac{1}{2} \cdot \frac{1}{z-3}$. We write each of these terms as geometric series in each region.

$$\frac{1}{z-1} = -\frac{1}{1-z} = -(1 + z + z^2 + \dots) \qquad \text{(converges for } |z| < 1)$$
$$\frac{1}{z-1} = \frac{1}{z(1-1/z)} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right) \qquad \text{(converges for } |z| > 1)$$
$$\frac{1}{z-3} = -\frac{1}{3(1-z/3)} = -(1 + (z/3) + (z/3)^2 + \dots) \qquad \text{(converges for } |z| < 3)$$
$$\frac{1}{z-3} = \frac{1}{z(1-3/z)} = \frac{1}{z}\left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \dots\right) \qquad \text{(converges for } |z| > 3)$$

On each region we can add the appropriate form of these series.

On A_1 : $|z| < 1$:

$$f(z) = \frac{1}{2}\left(1 + z + z^2 + \ldots\right) - \frac{1}{2}\cdot\frac{1}{3}\left(1 + z/3 + (z/3)^2 + \ldots\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(1 - \frac{1}{3^{n+1}}\right)z^n.$$

On A_2 : $1 < |z| < 3$:

$$f(z) = -\frac{1}{2}\cdot\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right) - \frac{1}{6}\left(1 + z/3 + (z/3)^2 + \ldots\right) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{z^n} - \frac{1}{6}\sum_{n=0}^{\infty}(z/3)^n.$$

On A_3 : $|z| > 3$:

$$f(z) = -\frac{1}{2}\cdot\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right) + \frac{1}{2z}\left(1 + \frac{3}{z} + \frac{3^3}{z^2} + \ldots\right) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1 + 3^{n-1})}{z^n}.$$

Problem 3.

Find the first few terms of the Laurent series around 0 for the following.

(a) $f(z) = z^2\cos(1/3z)$ *for* $0 < |z|$.

Solution: Using the known series for $\cos(z)$ we get

$$f(z) = z^2\left(1 - \frac{1}{2!\cdot 3^2 z^2} + \frac{1}{4!\cdot 3^4 z^4} - \ldots\right) = z^2 - \frac{1}{2!\cdot 3^2} + \frac{1}{4!\cdot 3^4 z^2} - \ldots$$

(b) $f(z) = \dfrac{1}{e^z - 1}$ *for* $0 < |z| < R$. *What is* R*?*

Solution: Writing out e^z as a power series we have

$$f(z) = \frac{1}{z + z^2/2! + z^3/3! + \ldots} = \frac{1}{z}\cdot\frac{1}{1 + z/2! + z^2/3! + \ldots}$$

For z near 0 the expression $z/2! + z^2/3! + \ldots$ is small so we can use the geometric series:

$$f(z) = \frac{1}{z}\left(1 - (z/2! + z^2/3! + \ldots) + (z/2! + z^2/3! + \ldots)^2 - (z/2! + z^2/3! + \ldots)^3 + \ldots\right).$$

It is hard to get a general expression for the terms of this series, but we can compute the first few explicitly.

$$f(z) = \frac{1}{z}\left(1 - \frac{z}{2} + z^2(-1/3! + 1/4) + z^3(-1/4! + 2/(2!3!) - 1/8)\right) = \frac{1}{z}\left(1 - \frac{z}{2} + \frac{z^2}{12} - 0\cdot z^3 + \ldots\right)$$

Problem 4.

What is the annulus of convergence for $\displaystyle\sum_{n=-\infty}^{\infty}\frac{z^n}{2^{|n|}}$.

Solution: We find region for singular and regular parts seperately.

Singular part: $\sum_{n=1}^{\infty} \frac{1}{2^n z^n}$. Either by recognizing this as a geometric series or using the ratio test we see it converges if $|z| > 1/2$.

Regular part: $\sum_{n=0}^{\infty} \frac{z^n}{2^n}$. Either by recognizing this as a geometric series or using the ratio test we see it converges if $|z| < 2$.

The annulus of convergence is $\boxed{1/2 < |z| < 2}$.

Problem 5.
Find and classify the isolated singularities of each of the following. Compute the residue at each such singularity.

(a) $f_1(z) = \frac{z^3 + 1}{z^2(z + 1)}$

Solution: f_1 has a pole of order 2 at $z = 0$ and a apparently a simple pole at $z = -1$. (In fact we will see that $z = -1$ is a removable singularity.)

Res$(f_1, 0)$: Let $g(z) = z^2 f_1(z) = \frac{z^3+1}{z+1}$. Clearly we want the coefficient of z in the Taylor series for g. That is $\boxed{\text{Res}(f_1, 0) = g'(0) = -1}$. (Alternatively we could have written $1/(z + 1)$ as a geometric series and found the coefficient of z from that.)

Res$(f_1, -1)$: Let $g(z) = (z + 1)f_1(z) = \frac{z^3+1}{z^2}$. Res$(f_1, -1) = g(-1) = 0$. So the singularity is removable. In retrospect we could have seen this because $z^{+}1 = (z + 1)(z^2 - z + 1)$.

(b) $f_2(z) = \frac{1}{e^z - 1}$

Solution: f_2 has poles whenever $e^z - 1 = 0$, i.e. when $z = 2n\pi i$ for any integer n. We'll show the poles are simple and compute their residues all at once by computing $\lim_{z\to 2n\pi i} (z - 2n\pi i)f_z(z)$.

$$\text{Res}(f, 2n\pi i) = \lim_{z\to 2n\pi i} (z - 2n\pi i)f_z(z) = \lim_{z\to 2n\pi i} \frac{z - 2n\pi i}{e^z - 1} = \frac{1}{e^{2n\pi i}} = 1.$$

(The limit was computed using L'Hospital's rule.) Since the limit exists the pole is simple and the limit is the residue.

(c) $f_3(z) = \cos(1 - 1/z)$

Solution: f_3 has exactly one singularity, which is at $z = 0$. We'll find the residue by computing the first few terms of the Laurent expansion.

$$\cos(1 - 1/z) = \frac{e^{i(1-1/z)} + e^{-i(1-1/z)}}{2} = \frac{e^i e^{-i/z} + e^{-i} e^{i/z}}{2}.$$

Using the power series for e^w we have

$$e^i e^{-i/z} = e^i \left(1 - \frac{i}{z} - \frac{1}{2z^2} + \dots\right)$$
$$e^{-i} e^{i/z} = e^{-i} \left(1 + \frac{i}{z} - \frac{1}{2z^2} + \dots\right)$$

Looking at just the $1/z$ terms we have

$$\text{Res}(f_3, 0) = \frac{-ie^i + ie^{-i}}{2} = \sin(1).$$

Alternatively we could have used the trig identity $\cos(1-1/z) = \cos(1)\cos(1/z) + \sin(1)\sin(1/z)$.

Problem 6.
(a) *Find a function f that has a pole of order 2 at $z = 1+i$ and essential singularies at $z = 0$ and $z = 1$.*

Solution: It's easiest to write this as a sum.

$$f(z) = e^{1/z} + e^{1/(z-1)} + \frac{1}{(z-1-i)^2}.$$

The term $e^{1/z}$ has an essential singularty at $z = 0$. Since the other two terms are analytic at $z = 1$, f has an essential singurity at $z = 0$.

The singularities at 1 and $1+i$ can be analyzed in the same manner.

(b) *Find a function f that has a removable singularity at $z = 0$, a pole of order 6 at $z = 1$ and an essential singularity at $z = i$.*

Solution: We'll do this in the same way as part (a).

$$f(z) = \frac{z^2 + 8z}{\sin(z)} + \frac{1}{(z-1)^6} + e^{1/(z-i)}.$$

Problem 7.
True or false. If true give an argument. If false give a counterexample

(a) *If f and g have a pole at z_0 then $f+g$ has a pole at z_0.*

(b) *If f and g have a pole at z_0 and both have nonzero residues the fg has a pole at z_0 with a nonzero residue.*

(c) *If f has an essential singularity at $z = 0$ and g has a pole of finite order at $z = 0$ the $f+g$ has an essential singularity at $z = 0$.*

(d) *If $f(z)$ has a pole of order m at $z = 0$ then $f(z^2)$ has a pole of order $2m$*

Answers.(a) False. Counterexample: $f(z) = 1/z$, $g(z) = -1/z$.

(b) False. Counterexample: $f(z) = 1/z$, $g(z) = 1/z$.

(c) True. When you add Laurent series you simply add the coefficients. The singular part of the series for f has infinitely many nonzero coefficients. After a certain point, the singular part of g has all zero coefficients. So after that point, the singular part of $f+g$ has the same coefficients as f. That is, it has infinitely many nonzero coefficients, so the singularity is essential.

(d) True. We know $f(z) = z^{-m}g(z)$, where $g(0) \neq 0$. So, $f(z)^2 = z^{-2m}g(z^2)$, where $g(0^2) \neq 0$. This shows, $f(z^2)$ has a 0 of order $2m$.

Problem 8.
Find the Laurent series for each of the following.

(a) $1/\mathrm{e}^{(1-z)}$ *for* $1 < |z|$.

Solution: $f(z) = 1/\mathrm{e}^{(1-z)} = \mathrm{e}^{z-1}$ is analytic on the entire plane. So,

$$f(z) = \mathrm{e}^{-1}\mathrm{e}^{z} = \mathrm{e}^{-1}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)$$

is the Taylor series for all z. Hence it is the Laurent series on $|z| > 1$.

Problem 9.
Let $h(z) = \dfrac{1}{\sin(z)} - \dfrac{1}{z} + \dfrac{2z}{z^2 - \pi^2}$ *in the disk* $|z| < 2\pi$.

(a) *Show that all the apparent singularities are removable.*

(b) *Find the first 4 terms of the Taylor series around* $z = 0$.

Answers.(a) The apparent singularities of h are at $0, \pm\pi$. There might be a slicker way to do this part, but here's one that's not too painful.

At $z = 0$: $h(z) = \dfrac{z - \sin(z)}{z\sin(z)} + \dfrac{2z}{z^2 - \pi^2}$. The second term is analytic, so doesn't contribute to the singularity at 0. Writing out the first term in terms of Taylor series we have

$$\frac{z - \sin(z)}{z\sin(z)} = \frac{z^3/3! - z^5/5! + \ldots}{z^2 - z^4/3! + \ldots}$$

The numerator has a zero of order 3 and the denominator one of order 2, so the entire term has a 0 of order 1, i.e. the singularity is removable.

We can play the same game at $z = \pi$. To make things easier we use partial fractions

$$h(z) = \frac{1}{\sin(z)} - \frac{1}{z} + \frac{1}{z - \pi} + \frac{1}{z + \pi} = \frac{(z - \pi) + \sin(z)}{(z - \pi)\sin(z)} - \frac{1}{z} + \frac{1}{z + \pi}.$$

The second and third terms are analytic at $z = \pi$, so don't contribute to the singularity. The first term can be written as

$$\frac{(z-\pi) + \sin(z)}{(z-\pi)\sin(z)} = \frac{(z-\pi) + \left(-(z-\pi) + (z-\pi)^3/3! - (z-\pi)^5/5! + \ldots\right)}{(z-\pi)\left(-(z-\pi) + (z-\pi)^3/3! - (z-\pi)^5/5! + \ldots\right)} = \frac{(z-\pi)^3/3!}{(z-\pi)^2(-1 + (z-\pi)^2/2 + \ldots)}$$

As before, the numerator has a zero of order 3 and the denominator one of order 2, so the singularity is removable.

The singularity at $z = -\pi$ is handled identically to $z = \pi$.

(b) For this part let's work on each term in the original expression of h.

$$\begin{aligned}
\frac{1}{\sin(z)} &= \frac{1}{z}\frac{1}{1 - (z^2/3! - z^4/5! + \ldots)} \\
&= \frac{1}{z}\left(1 + (z^2/3! - z^4/5! + \ldots) + (z^2/3! - z^4/5! + \ldots)^2 + \ldots\right) \\
&= \frac{1}{z}\left(1 + z^2/3! + z^4(-1/5! + 1/(3!)^2) + \ldots\right) \\
&= \frac{1}{z}\left(1 + \frac{z^2}{6} + \frac{7z^4}{360} + \ldots\right)
\end{aligned}$$

$$\frac{2z}{z^2-\pi^2} = -\frac{2z}{\pi^2(1-z^2/\pi^2)} = -\frac{2z}{\pi^2}(1+z^2/\pi^2+z^4/\pi^4+\ldots)$$

Combining all the parts we get

$$\begin{aligned} h(z) &= \frac{1}{z}+\frac{z}{6}+\frac{7z^3}{360}+\ldots-\frac{1}{z}-\frac{2z}{\pi^2}-\frac{2z^3}{\pi^4}+\ldots \\ &= 0+z\left(\frac{1}{6}-\frac{2}{\pi^2}\right)+z^3\left(\frac{7}{360}-\frac{2}{\pi^4}\right)+\ldots \end{aligned}$$

Problem 10.

Find the residue at ∞ of each of the following.

(a) $f(z) = e^z$

(b) $f(z) = \dfrac{z-1}{z+1}$.

Answers.(a) Easy method: $f(z)$ is entire so $\int_C f(z)\,dz = 0$ for all closed C. Since the residue at infinity is minus the integral over a closed curve containing all the singularities we must have $\text{Res}(f,\infty) = 0$.

Method 2. Let $g(w) = \frac{1}{w^2}e^{1/w}$. The Laurent series for g is

$$g(w) = \frac{1}{w^2}\left(1+\frac{1}{w}+\ldots\right).$$

So $\text{Res}(f,\infty) = -\text{Res}(g,0) = 0$.

(b) Since Let $g(w) = \dfrac{1}{w^2}f(/1/w) = \dfrac{1}{w^2}\dfrac{1/w-1}{1/w+1}$. Writing $1/(w+1)$ as a geometric series we get

$$g(w) = \frac{1}{w^2}(1-w)(1-w+w^2-w^3+\ldots) = \frac{1}{w^2}(1-2w+2w^2-\ldots) = \frac{1}{w^2}-\frac{2}{w}+2-\ldots$$

Therefore $\text{Res}(f,\infty) = -\text{Res}(g,0) = 2$.

Problem 11.

Use the following steps to sketch the stream lines for the flow with complex potential $\Phi(z) = z+\log(z-i)+\log(z+i)$

(i) Identify the components, i.e. sources, sinks, etc of the flow.

(ii) Find the stagnation points.

(iii) Sketch the flow near each of the sources.

(iv) Sketch the flow far from the sources.

(v) Tie the picture together.

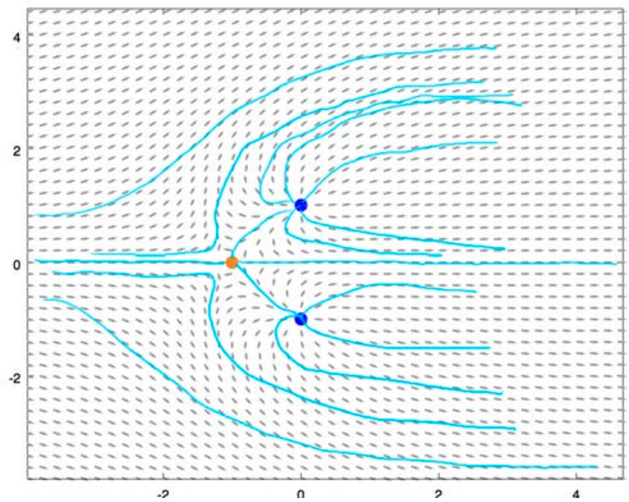
Solution: (i) The log terms with positive coefficients represent sources. The term z represents a steady stream. So this is two sources in a steady stream.

(ii) Stagnation poihts are places where $\Phi'(z) = 0$. Computing:

$$\Phi'(z) = 1 + \frac{2z}{z^2+1} = \frac{(z+1)^2}{z^2+1}.$$

So there is a single stagnation point at $z = -1$.

(iii-v) Near the sources the flow looks like a source. Far away it looks like uniform flow to the right. By symmetry (or direct calculation) there are streamlines on the x-axis. We get the following picture. (I used Octave to draw draw the underlying vector field.)



Problem 12.

Compute the following definite integrals

(a) $\displaystyle\int_{-\pi}^{\pi} \frac{1}{1+\sin^2(\theta)}\, d\theta$. *(Solution: $\pi\sqrt{2}$)*

Solution: On the unit circle $z = e^{i\theta}$, $\sin(\theta) = \dfrac{z - 1/z}{2i} = \dfrac{z^2-1}{21}$. So the integral becomes

$$\int_{|z|=1} \frac{1}{1 + ((z^2-1)/2iz)^2}\,\frac{dz}{iz} = \int_{|z|=1} \frac{-4z}{i(z^4 - 6z^2 + 1)}\,dz$$

Let $f(z) = \dfrac{-4z}{i(z^4 - 6z^2 + 1)}$. So the integral is

$$\int_{|z|=1} f(z)\,dz = 2\pi i \sum \text{residues of } f \text{ inside the unit disk.}$$

The poles of f are at $z^2 = 3 \pm \sqrt{8}$. Of these, only $z^2 = 3 - \sqrt{8}$ is inside the unit circle. So there are two poles inside the unit circle at $z_1 = \sqrt{3 - \sqrt{8}}$ and $z_2 = -\sqrt{3-\sqrt{8}}$. These are simple poles and we can compute the residue using L'Hospital's rule.

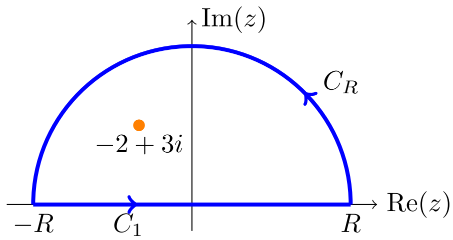
$$\text{Res}(f, z_1) = \lim_{z \to z_1} \frac{(z - z_1)(-4z)}{i(z^4 - 6z^2 + 1)} = \frac{-4z_1}{i(4z_1^3 - 12z_1)} = \frac{-1}{i(z_1^2 - 3)} = \frac{1}{i\sqrt{8}}.$$

The residue at z_2 has the same value. So,

$$\int_{|z|=1} f(z)\,dz = 2\pi i(\mathrm{Res}(f,z_1)+\mathrm{Res}(f,z_2)) = \frac{4\pi}{\sqrt{8}} = \pi\sqrt{2}.$$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2}\,dx$. *(Solution: $-\pi/27$)*

Solution: Call the integral in question I. Let $f(z) = z/(z^2+4z+13)^2$. This decays faster than $1/z^2$ so we can use path



We know $\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0$, so, letting R go to infinity in $\displaystyle\int_{C_1+C_R} f(z)\,dz$ we get

$$I = 2\pi i \sum \text{ residues of } f \text{ in the upper half-plane.}$$

The poles of f are at $-2\pm 3i$. Only $z_1 = -2+3i$ is in the upper half-plane. All we have to do is compute the residue. Let $g(z) = (z-z_1)^2 f(z) = \frac{z}{(z-(-2-3i))^2}$. Since g is analytic at z_1 we have

$$\mathrm{Res}(f,z_1) = g'(z_1) = \text{ some algebra } = i/54.$$

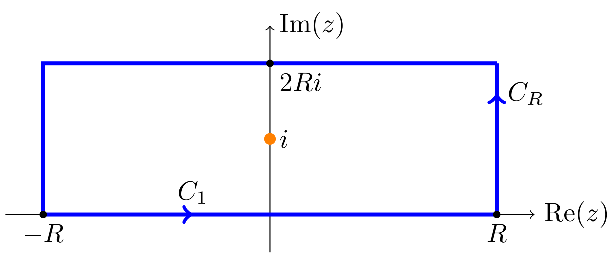
So $I == -\pi/27$.

(c) *p.v.* $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(x)}{1+x^2}\,dx$.

Solution: Call the integral in question I. Replace $\sin(x)$ by e^{ix} and let

$$\tilde{I} = \text{p.v.}\int_{-\infty}^{\infty} \frac{xe^{ix}}{1+x^2}\,dx, \qquad \text{so, } I = \mathrm{Im}(\tilde{I})$$

Let $f(z) = \dfrac{ze^{iz}}{1+z^2}$ and use the contour $C_1 + C_R$.



The only pole of f in the upper half-plane is at $z = i$. It is easy to compute $\mathrm{Res}(f,i) = ie^{-1}2i = e^{-1}/2$. So,

$$\int_{C_1+C_R} f(z)\,dz = 2\pi i\,\mathrm{Res}(f,i) = \pi i e^{-1}.$$

Since $|z/(1+z^2)| < M/|z|$ for Large z and the coefficient of ix in the exponent of f is positive, we know

$$\lim_{R\to\infty} f(z)\,dz = 0.$$

Also, $\lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = \text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \tilde{I}$.
In conclusion we have

$$\tilde{I} = 2\pi i \operatorname{Res}(f, i) = \pi i e^{-1}.$$

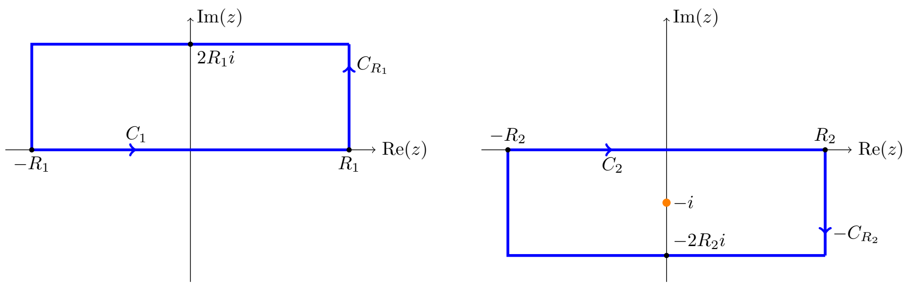
So $I = \operatorname{Im}(\tilde{I}) = \boxed{\pi e^{-1}.}$

(d) *p.v.* $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{x+i}\,dx.$

Solution: Write $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$. So, Let $f_1(z) = \frac{e^{iz}}{z+i}$ and $f_2(z) = \frac{e^{-iz}}{z+i}$.

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{cos(x)}{x+i}\,dx = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{2} f_1(x)\,dx + \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{2} f_2(x)\,dx.$$

We compute these integrals using two different contours



The reasoning is the same as in part (b). Both f_1 and f_2 have a single pole at $z - i$. So, using the contour $C_1 + C_{R_1}$ we find

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{1}{2} f_1(x)\,dx = 2\pi i \operatorname{Res} \frac{1}{2} f_1 \text{ in the upper half plane.} = 0.$$

Likewise, using the contour $C_1 - C_{R_2}$ we find

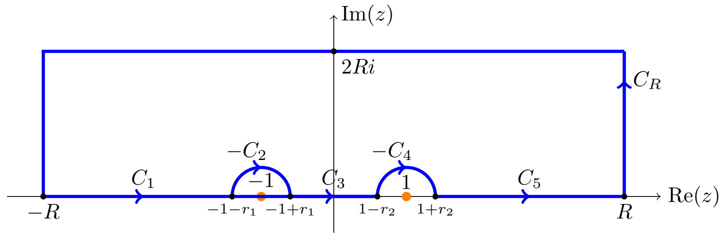
$$\text{p.v.} \int_{-\infty}^{\infty} \frac{1}{2} f_2(x)\,dx = 2\pi i \operatorname{Res} \frac{1}{2} f_2 \text{ in the lower half plane.} = -2\pi i \operatorname{Res}(f_2/2, -i) = -\pi i e^{-1}.$$

(The minus sign is because $C_1 - C_{R_2}$ is oriented in the clockwise direction.)

Answer to problem: the integral is $-\pi i e^{-1}$.

(e) $I =$ *p.v.* $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{2ix}}{x^2-1}\,dx.$

Solution: Since our integrand $f(z) = \dfrac{ze^{2iz}}{z^2-1}$ has poles on the real axis we will need to use an indented contour.



As usual, we chose the contour so that the integral over C_R goes to 0 as R goes to infinity. Since f has no poles inside the contour we have

$$\int_{C_1-C_2+C_3-C_4+C_5+C_R} f(z)\,dz = 0.$$

The poles of f at ± 1 are simple. So, letting $R \to \infty$ and $r_1, r_2 \to 0$ we get

$$I = \pi i(\text{Res}(f,-1) + \text{Res}(f,1)).$$

The residues are straightforward to compute.

$$\text{Res}(f,-1) = \text{e}^{-2i}/2, \qquad \text{Res}(f,1) = \text{e}^{2i}/2.$$

So, $I = \pi i(\text{e}^{2i} + \text{e}^{-2i})/2 = \boxed{\pi i \cos(2)}$.
