18.04 Problem Set 1, Spring 2018

Calendar
W Feb. 7: Reading: Topic 0, Topic 1 sections 1.1-1.5.2
R Feb. 8:
F Feb. 9: Reading: Topic 1 sections 1.5.3-end

Coming next

Feb. 12-16: Analytic functions

Problem 1. (30: 10,5,10,5 points)
(a) Let $z_1 = 1 + i$, $z_2 = 1 + 3i$. Compute $z_1 z_2$, z_1/z_2, $z_1^{z_2}$ (use the principal branch of log).

(Give $z_1 z_2$ and z_1/z_2 in standard rectangular form and $z_1^{z_2}$ in polar form.)

(b) Compute all the values of i^i. Say which one comes from the principal branch of log.

(Give all your answers in standard form.)

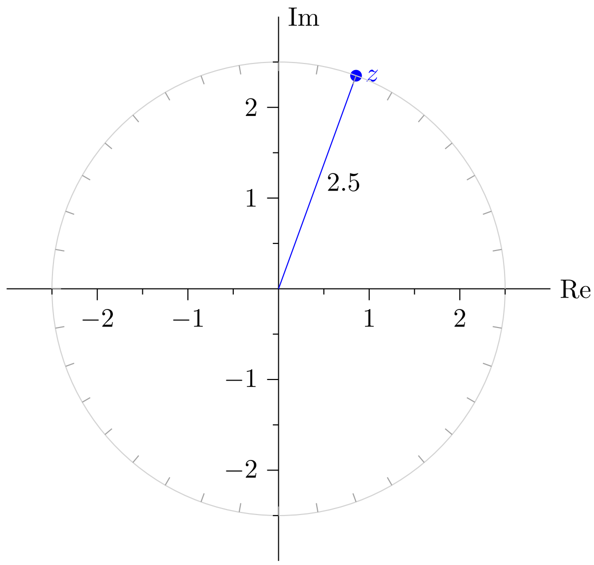
Is it surprising that i^i is real?

(c) Let $z = 1 + i\sqrt{3}$.

(i) Compute z^8. *(Give your answer in standard form.)*

(ii) Find all the 4th roots of z.

(d) Copy the following figure and add all the 5th roots of z to it. (The figure indicates that $|z| = 2.5$. The circle on the outside is a handy protractor marked off in 10° increments.)



Problem 2. (15: 5,5,5 points)
(a) Show $\overline{e^z} = e^{\overline{z}}$.

(b) Show that if $|z| = 1$ then $z^{-1} = \overline{z}$.

(c) Let $\frac{x+iy}{x-iy} = a + ib$. Show that $a^2 + b^2 = 1$.

Hint: This takes one line if you look at it right. Think polar form.

Problem 3. (15: 5,10 points)
(a) Sketch the curve $z = \mathrm{e}^{t(1+i)}$, where $-\infty < t < \infty$.

(b) Consider the mapping $z \to w = z^2$. Draw the image in the w-plane of the triangular region in the z-plane with vertices 0, 1 and i.

Problem 4. (10 points)
Let $z_k = \mathrm{e}^{2\pi i/n}$. Show

$$1 + z_k + z_k^2 + z_k^3 + ... + z_k^{n-1} = 0$$

Hint: The polynomial $z^n - 1$ has one easy root. Use that to factor it into a linear term and a degree $n-1$ term.

Problem 5. (20: 10,10 points) (Orthogonal lines stay orthogonal!)
(a) Consider the mapping $w = \mathrm{e}^z$.
(i) Sketch in the w-plane the image under this mapping of vertical lines in the z-plane.

(ii) On the same graph sketch the image of horizontal lines.

Show enough lines to give a good idea of what's happening.

(iii) Show (argue either geometrically or analytically) that the images of a vertical and a horizontal lines meet at right angles.

(b) Repeat part (a) for the mapping $w = z^2$.

Extra problems not for points

Problem 6. (0 points) Find all points where

$$\mathrm{Arg}\left(\frac{z-1}{z+2}\right) = \pm\frac{\pi}{2}$$

Hint: let $w = (z-1)/z+2)$. What does the condition say about the relation between w and $\overline{w}$? Be careful to note points where $\mathrm{Arg}(w)$ is not defined.