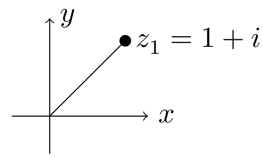
18.04 Problem Set 1, Spring 2018 Solutions

Problem 1. (30: 10,5,10,5 points)

(a) *Let $z_1 = 1+i$, $z_2 = 1+3i$. Compute $z_1 z_2$, z_1/z_2, $z_1^{z_2}$ (use the principal branch of log). (Give $z_1 z_2$ and z_1/z_2 in standard rectangular form and $z_1^{z_2}$ in polar form.)*

Solution: $z_1 = 1+i$, $z_2 = 1+3i$: $\boxed{z_1 z_2 = -2+4i}$, $\dfrac{z_1}{z_2} = \dfrac{1+i}{1+3i} \cdot \dfrac{1-3i}{1-3i} = \dfrac{4-2i}{10} = \boxed{\dfrac{2-i}{5}}$.



$|z_1| = \sqrt{2}$, $\operatorname{Arg}(z_1) = \pi/4$ (principal branch).
So $\log(z_1) = \log(\sqrt{2}) + i\frac{\pi}{4} = \frac{1}{2}\log(2) + i\frac{\pi}{4}$ (principal branch).
Thus,

$$z_1^{z_2} = \mathrm{e}^{z_2 \log(z_1)} = \mathrm{e}^{(1+3i)(\log(2)/2 + i\pi/4)} = \boxed{\mathrm{e}^{\log(2)/2 - 3\pi/4} \mathrm{e}^{i(3\log(2)/2 + \pi/4)}} \text{ (not pretty!)}$$

(b) *Compute all the values of i^i. Say which one comes from the principal branch of log. (Give all your answers in standard form.)*

Is it surprising that i^i is real?

Solution: $|i| = 1$ and $\operatorname{Arg}(i) = \pi/2$, so $\log(i) = \dfrac{\pi}{2}i + 2n\pi i$, where n is any integer. Thus

$$\boxed{i^i = \mathrm{e}^{i\log(i)} = \mathrm{e}^{-\pi/2 - 2n\pi}}$$

On the principal branch $i^i = \mathrm{e}^{-\pi/2}$. I'm surprised that it's real!

(c) *Let $z = 1 + i\sqrt{3}$.*

(i) Compute z^8. (Give your answer in standard form.)

(ii) Find all the 4th roots of z.

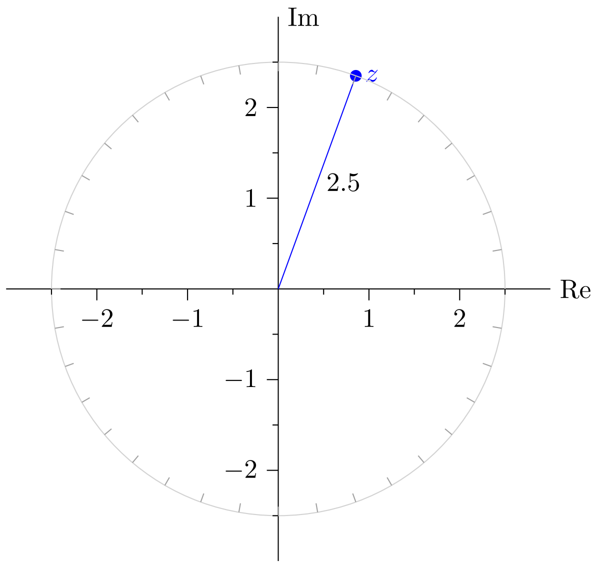
Solution: (i) $z = 1 + \sqrt{3}\,i = 2\mathrm{e}^{i\pi/3}$, so

$$z^8 = 256\mathrm{e}^{i8\pi/3} = 256\mathrm{e}^{i2\pi/3} = 128(-1 + \sqrt{3}\,i)$$

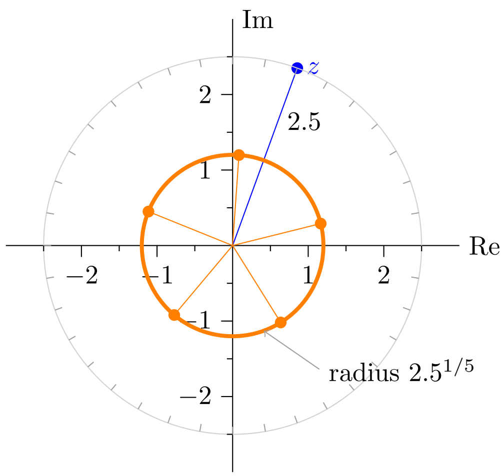
(ii) $z^{1/4} = (2\mathrm{e}^{i\pi/3 + 2n\pi i})^{1/4} = 2^{1/4}\mathrm{e}^{i\pi/12}$, $2^{1/4}\mathrm{e}^{i7\pi/12}$, $2^{1/4}\mathrm{e}^{i13\pi/12}$, $2^{1/4}\mathrm{e}^{i19\pi/12}$

Since $\pi/12 = 15^\circ$ this doesn't have a pretty standard form, so we'll leave them as is. We can always get numerical approximations if needed.

(d) *Copy the following figure and add all the 5th roots of z to it. (The figure indicates that $|z| = 2.5$. The circle on the outside is a handy protractor marked off in 10° increments.)*



Solution: We have $\arg(z) = 70^\circ = 7\pi/18$, so $z = 2.5\mathrm{e}^{i7\pi/18}$. The roots are shown in orange. The root in the first quadrant closest to the x-axis has argument $\arg(z)/5 = 7\pi/90 = 14^\circ$. The others are spaced at intervals of $2\pi/5$ (72°) around the circle of radius $|z|^{1/5} = (2.5)^{1/5}$.



Problem 2. (15: 5,5,5 points)
(a) *Show $\overline{\mathrm{e}^z} = \mathrm{e}^{\overline{z}}$.*

Solution: Let $z = x + iy$, then $\mathrm{e}^z = \mathrm{e}^x\mathrm{e}^{iy} = \mathrm{e}^x\cos(y) + i\mathrm{e}^x\sin(y)$. So,

$$\overline{(\mathrm{e}^z)} = \mathrm{e}^x\cos(y) - i\mathrm{e}^x\sin(y) = \mathrm{e}^x\mathrm{e}^{-iy} = \mathrm{e}^{x-iy} = \mathrm{e}^{\overline{z}} \quad \blacksquare$$

(b) *Show that if $|z| = 1$ then $z^{-1} = \overline{z}$.*

Solution: We'll give two ways to say this:
Method 1: $\dfrac{1}{z} = \dfrac{\overline{z}}{z\cdot\overline{z}} = \dfrac{\overline{z}}{|z|^2} = \dfrac{\overline{z}}{1} = \overline{z}$.

Method 2: $z = \mathrm{e}^{i\theta}$, so by part (a) $\overline{z} = \mathrm{e}^{-i\theta} = z^{-1}$.

(c) *Let $\frac{x+iy}{x-iy} = a + ib$. Show that $a^2 + b^2 = 1$.*

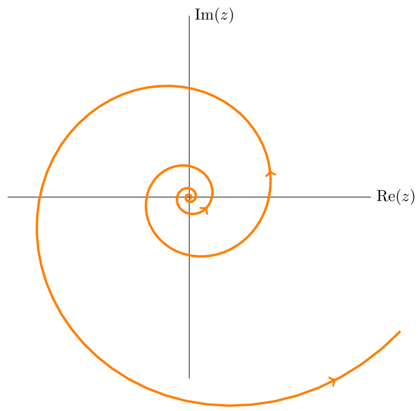
Hint: This takes one line if you look at it right. Think polar form.

Solution: If $x + iy = re^{i\theta}$, then $x - iy = re^{-i\theta}$. So, $\left|\frac{x+iy}{x-iy}\right| = \left|\frac{re^{i\theta}}{re^{-i\theta}}\right| = |e^{2i\theta}| = 1$. QED

Problem 3. (15: 5,10 points)

(a) *Sketch the curve $z = e^{t(1+i)}$, where $-\infty < t < \infty$.*

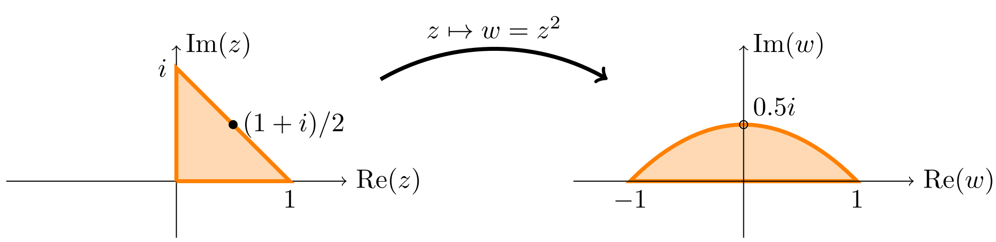
Solution: $e^{t(1+i)} = e^t e^{it}$. So as t grow the magnitude grows exponentially and the argument increases. This gives a spiral!



Note, this is not to scale. The exponential e^t grows too quickly, so we drew $e^{0.2t}$ instead. Qualitatively this is a good representation of the curve.

(b) *Consider the mapping $z \to w = z^2$. Draw the image in the w-plane of the triangular region in the z-plane with vertices 0, 1 and i.*

Solution: The figure shows the image.
In brief we have: $z = 1 \mapsto w = 1$, $z = (1 + i)/2 \mapsto w = i/2$, $z = i \mapsto w = -1$.
The sides of the triangle map as follows:
the line segment $[0, 1] \mapsto$ the line segment $[0, 1]$
the line segment $[0, i] \mapsto$ the line segment $[0, -1]$
the line segment $[1, i] \mapsto$ the parabola shown in the figure (proved below).



It takes a small amount of algebra to find the equation for the image of the segment from 1 to i. We parametrize the segment as $z = (1 - t) + ti$ with $0 \le t \le 1$. Then $w = z^2 = (1 - 2t) + 2t(1 - t)i$. As a curve in the plane this is

$$w = (u(t), v(t)), \text{ where } u(t) = 1 - 2t \text{ and } v(t) = 2t(1 - t).$$

A little more algebra shows that $t = (1 - u)/2$ and therefore $v = -u^2 + 1/2$. This is the formula for the parabola shown.

Problem 4. (10 points)
Let $z_k = \mathrm{e}^{2\pi i/n}$. Show

$$1 + z_k + z_k^2 + z_k^3 + ... + z_k^{n-1} = 0$$

Hint: The polynomial $z^n - 1$ has one easy root. Use that to factor it into a linear term and a degree $n-1$ term.

Solution: The nth roots of 1 are all the roots of the equation $z^n - 1 = 0$. Since we know that $z = 1$ is a root we can factor the equation as

$$(z-1)(z^{n-1} + z^{n-2} + ... + z + 1) = 0$$

Since z_k is also a root of $z^n - 1$ and $z_k \neq 1$, we must have z_k is a root of the second factor. This is what we were asked to show!

Problem 5. (20: 10,10 points) (Orthogonal lines stay orthogonal!)
(a) *Consider the mapping $w = \mathrm{e}^z$.*
(i) Sketch in the w-plane the image under this mapping of vertical lines in the z-plane.

(ii) On the same graph sketch the image of horizontal lines.

Show enough lines to give a good idea of what's happening.

(iii) Show (argue either geometrically or analytically) that the images of a vertical and a horizontal lines meet at right angles.

Solution: (i) As usual, let $z = x + iy$. Vertical line have the equation $x = a$ for some constant a. We can parametrize this line by $z = a + iy$, with $-\infty < y < \infty$. The image of the line under the mapping is

$$w = \mathrm{e}^{a+iy} = \mathrm{e}^a \mathrm{e}^{iy}.$$

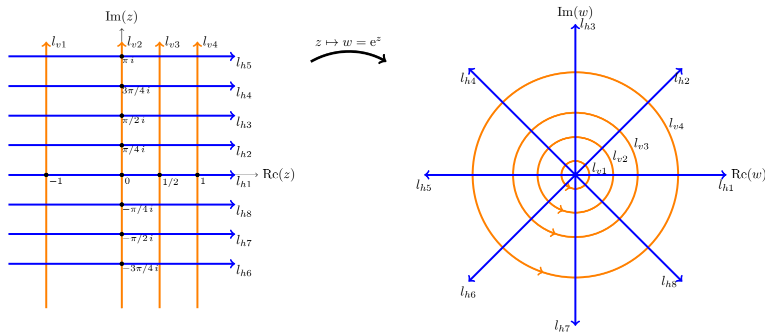
Thus $|w| = \mathrm{e}^a$ is constant, so the image is a circle of radius e^a centered on 0.

(ii) Likewise, horizontal lines have the equation $y = b$ for some constant b. We parametrize this by $z = x + ib$, with $-\infty < x < \infty$. The image of the line under this mapping is

$$w = \mathrm{e}^{x+ib} = \mathrm{e}^x \mathrm{e}^{ib}.$$

Thus $\arg(w) = b$ is constant, so the image is a ray from the origin at angle b.

In the figure below we label the corresponding curves with the same label. So, for example, the image of the vertical line l_{v4} is the circle in the w-plane labeled l_{v4}



(iii) The images found in parts (i) and (ii) are circles and rays. Geometrically we know that radii are perpendicular to circles, so rays from the origin are perpendicular to circles centered on the origin.

(b) *Repeat part (a) for the mapping $w = z^2$.*

Solution: This part is similar to part (a), though we will need a little more algebraic manipulation to identify the image curves.

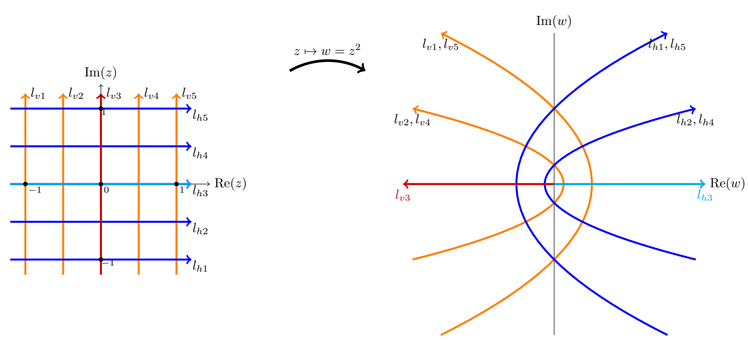
(i) The vertical line $x = a$ is parametrized by $z = a + iy$, with $-\infty < y < \infty$. So the image is $w = a^2 - y^2 + i2ay$. Let's write this as $w = u + iv$. It is easy to see that with $u = a^2 - y^2$ and $v = 2ay$ we have

$$u = a^2 - (v/2a)^2.$$

This is a parabola in the uv-plane which opens to the left and has vertex at $(a^2, 0)$.

(ii) Horizontal lines $y = b$ are similar:
Parametrization: $z = x + ib$, with $-\infty < x < \infty$.
Image: $w = x^2 - b^2 + i2bx$.
If $w = u + iv$ then u $= (v/2b)^2 - b^2$. This is a parabola opening to the right with vertex at $(-b^2, 0)$.

The figure below is labeled in the same way as the figure in part (a). Notice the vertical lines $x = a$ and $x = -a$ map to the same parabola as befits the 2-to-1 mapping $w = z^2$. The real z-axis ($x = 0$) maps to the positive real w-axis ($u > 0$). We can view this as a degenerate parabola. Likewise for the imaginary z-axis.



(iii) To show the image curves are orthogonal (meet at right angles) we need to show that their tangent vectors are orthogonal. Happily, we've parametrized the image curves in parts (i) and (ii), so computing tangent vectors is easy. In keeping with the course we will write vectors as $x + iy$ rather than in 18.02 notation as $\langle x, y \rangle$

Horizontal line: $z = a+iy$; image $w = a^2-y^2+i2ay$; image tangent vector: $\frac{dw}{dy} = -2y+i2a$.
Vertical line: $z = x+ib$; image $w = x^2-b^2+i2xb$; image tangent vector: $\frac{dw}{dx} = 2x+i2b$.

The lines meet at the point $a+ib$, so we need to check the tangents at that point.
Horizontal line image: $\frac{dw}{dy} = -2b+i2a$
Vertical line image: $\frac{dw}{dx} = 2a+i2b$
It is easy to see that these are orthogonal. For 18.04 the nicest way to say this is to note that $-2b+i2a = i(2a+i2b)$. Since multiplication by i is rotation by 90° the vectors are at right angles.

Extra problems not for points

Problem 6. (0 points) *Find all points where*

$$\mathrm{Arg}\left(\frac{z-1}{z+2}\right) = \pm\frac{\pi}{2}$$

Hint: let $w = (z-1)/z+2)$. What does the condition say about the relation between w and $\overline{w}$? Be careful to note points where $\mathrm{Arg}(w)$ is not defined.

Solution: Answer: the set of points is the circle $|z+1/2| = 3/2$, excluding the points $z = 1$ and $z = -2$ where $\arg(w)$ is not defined.

Justification. $\mathrm{Arg}(w) = \pm\pi/2$ means that w is on the imaginary axis (and $w \neq 0$). Thus we must have

$$w = \frac{z-1}{z+2} = ib, \text{ where } b \neq 0 \text{ is real.}$$

Solving for z we get $z = \dfrac{2ib+1}{-ib+1}$.

We can view this as a mapping $b \mapsto z$ from the real b-line to the complex z-plane. A tiny amount of algebra gives

$$z = \frac{2ib+1}{-ib+1} = \frac{3}{-ib+1} - 2.$$

Ignoring the scale by 3 and the shift by -2 for the moment we have:

Claim. The image of $w = \frac{1}{-ib+1}$ where b is real is the circle $|w-1/2| = 1/2$, i.e. the circle with radius 1/2 and center 1/2.

Proof.

$$|w-1/2| = \left|\frac{1}{-ib+1} - 1/2\right| = \left|\frac{1+bi}{2(1-bi)}\right| = \frac{1}{2}. \qquad \blacksquare$$

(The last equality follow from problem 2c.)

Since $z = 3w-2$ we have the z-image is the circle of radius 3/2 with center $3/2-2 = -1/2$.

Alternative solution. $w = (z-1)/(z+2)$ is pure imaginary, so $w = -\overline{w}$, i.e. $\dfrac{z-1}{z+2} = -\dfrac{\overline{z}-1}{\overline{z}+2}$. A little algebra gives $2|z|^2 + z + \overline{z} = 4$.

Let $z = x+iy$. The above becomes $2(x^2+y^2)+2x = 4$ or $x^2+y^2+x = 4$. Completing the square gives $(x+1/2)^2+y^2 = 9/4$. That is $|z+1/2|^2 = 9/4$.