# 18.04 Problem Set 2, Spring 2018 Solutions

**Problem 1.** (20: 10,10 points)
**(a)** *Show that $\cos(z)$ is an analytic for all $z$, i.e. it's an entire function. Compute its derivative and show it equals $-\sin(z)$.*

We'll do this two ways, first from the definition of $\cos(z)$ in terms of exponentials. Second, we'll write $\cos(z)$ as a function of $x$ and $y$ and verify the Cauchy-Riemann equations.

**Method 1.** By definition $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$, so $\cos(z)$ is entire because both $e^{iz}$ and $e^{-iz}$ are entire. Its derivative is

$$\frac{d\cos(z)}{dz} = \frac{ie^{iz} - ie^{-iz}}{2} = \frac{-e^{iz} + e^{-iz}}{2i} = -\sin(z)$$

**Method 2.** First let's write $\cos(z) = u(x, y) + iv(x, y)$

$$\cos(z) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2} = \ldots = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

(You can work out the algebraic details of this formula.) Next we compute the partials and verify the Cauchy-Riemann equations.

$$\begin{aligned} u_x &= -\sin(x)\cosh(y) \qquad & u_y &= \cos(x)\sinh(y) \\ v_x &= -\cos(x)\sinh(y) \qquad & v_y &= -i\sin(x)\cosh(y) \end{aligned}$$

Since $u_x = v_y$ and $u_y = v_x$ the Cauchy-Riemann equations hold for all $z$ and $\cos(z)$ is entire and

$$\frac{d\cos(z)}{dz} = u_x - iu_y = -\sin(x)\cosh(y) - i\cos(x)\sinh(y) = -\sin(z).$$

(You can easily check that the last expression is indeed $-\sin(z)$.)

**(b)** *Give the region where $\cot(z)$ is analytic. Compute its derivative.*

Since $\cot(z) = \dfrac{\cos(z)}{\sin(z)}$ is the quotient of entire functions it is analytic for all $z$ except where $\sin(z) = 0$. We know $\sin(z) = 0$ for all multiples of $\pi$. To see that this is all the zeros of sin we use the formula

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

Since $\cosh(y)$ is never 0, the real part of $\sin(z)$ is only 0 where $\sin(x) = 0$, where $x = n\pi$ for some integer $n$. Since $\cos(n\pi) \neq 0$, the imaginary part of $\sin(z)$ is only 0 if $\sinh(y) = 0$. This only happens when $y = 0$ Therefore, the zeros of $\sin(z)$ are at $z = x + iy = n\pi$.

So $\cot(z)$ is analytic on the set $\mathbf{C} - \{n\pi \text{ where } n \text{ is an integer}\}$.

We use the quotient rule to compute the derivative. Since the algebra will be identical to the real case, we know the derivative will be $-\csc^2(z)$:

$$\frac{d\cot(z)}{dz} = \frac{d}{dz}\left(\frac{\cos(z)}{\sin(z)}\right) = \frac{-\sin(z)\sin(z) - \cos(z)\cos(z)}{\sin^2(z)} = -\frac{1}{\sin^2(z)} = -\csc^2(z).$$

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**Problem 2.** (20: 10,10 points)

**(a)** *Let* $P(z) = (z - r_1)(z - r_2)\ldots(z - r_n)$. *Show that* $\dfrac{P'(z)}{P(z)} = \displaystyle\sum_{j=1}^{n} \frac{1}{z - r_j}$

*Suggestion: try* $n = 2$ *and* $n = 3$ *first.*

Following the suggestion for $n = 2$: let $P(z) = (z - r_1)(z - r_2)$. Using the product rule for $P'$ we get

$$\frac{P'(z)}{P(z)} = \frac{(z - r_2) + (z - r_1)}{(z - r_1)(z - r_2)} = \frac{1}{z - r_1} + \frac{1}{z - r_2}.$$

This is exactly what was claimed. The only difficulty in going to larger $n$ is in presenting the argument. We'll let $(z - r_1)(z - r_2)\cancel{(z - r_3)}\ldots(z - r_n)$ mean the product of all the terms leaving out the one with the line throught it. Then if $P(z) = (z - r_1)(z - r_2)\ldots(z - r_n)$ the product rule gives us

$$P'(z) = \sum_{j=1}^{n} (z - r_1)(z - r_2)\cdots\cancel{(z - r_j)}\cdots(z - r_n).$$

From this it is clear that

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{(z - r_1)(z - r_2)\cdots\cancel{(z - r_k)}\cdots(z - r_n)}{(z - r_1)(z - r_2)\cdots(z - r_j)\cdots(z - r_n)} = \sum_{j=1}^{n} \frac{1}{z - r_j}$$

**(b)** *Compute and simplify* $\dfrac{d}{dz}\left(\dfrac{az + b}{cz + d}\right)$.

*What happens when* $ad - bc = 0$ *and why?*

Let $f(z) = \dfrac{az + b}{cz + d}$. The quotient rule gives

$$f'(z) = \frac{a(cz + d) - (az + b)c}{(cz + d)^2} = \boxed{\frac{ad - bc}{(cz + d)^2}}.$$

If $ad - bc = 0$ then the derivative is always 0, so $f(z)$ must be constant. We verify this directly: We know $a/c = b/d$, call this ratio $r$. Then

$$f(z) = \frac{az + b}{cz + d} = \frac{rcz + rd}{cz + d} = r.$$

This shows that $f(z)$ is constant.

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**Problem 3.** (10 points)

*Why does* $\log(e^z)$ *not always equal* $z$*?*

*Hint: This is true for any branch of* log. *Start with the principal branch.*

The function $e^z$ is many-to-one so it can't possibly have an inverse. For example, $e^0 = e^{2\pi i} = e^{4\pi i} = \ldots = 1$. So, on any branch of log we'll have

$$\log(e^0) = \log(e^{2\pi i}) = \log(1)$$

For example, if we choose the principal branch of log the $\log(1) = 0$, so $\log(e^{2\pi i}) \neq 2\pi i$.

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**Problem 4.** (20: 10,10 points)
**(a)** *Let $f(z)$ be analytic in a $D$ a disk centered at the origin. Show that $F_1(z) = \overline{f(\overline{z})}$ is analytic in $D$.*

**(b)** *Let $f(z)$ be as in part (a). Show that $F_2(z) = f(\overline{z})$ is not analytic unless $f$ is constant.*

*Hint for both parts: Use the Cauchy-Riemann equations.*

The tricky part of this problem is keeping the notation straight while we take partial derivatives for use in the Cauchy-Riemann equations. So, for $z = x + iy$, let's write

$$f(z) = u(x, y) + iv(x, y).$$

**(a)** Then

$$F_1(z) = \overline{f(\overline{z})} = u(x, -y) - iv(x, -y).$$

We can write $F_1(z) = U_1(x, y) + iV_1(x, y)$, where $U_1(x, y) = u(x, -y)$ and $V_1(x, y) = -v(x, -y)$. Since $f(z)$ is analytic the Cauchy-Riemann equations say that $u_x = v_y$ and $u_y = -v_x$. To check the Cauchy-Riemann equations on $F_1$ we take the partial derivatives of $U_1$ and $V_1$. (We need to be careful with the $-y$ when taking partials with respect to $y$.):

$$\begin{aligned}
\frac{\partial U_1}{\partial x}(x, y) &= \frac{\partial u}{\partial x}(x, -y), & \frac{\partial U_1}{\partial y} &= -\frac{\partial u}{\partial y}(x, -y) \\
\frac{\partial V_1}{\partial x}(x, y) &= -\frac{\partial v}{\partial x}(x, -y), & \frac{\partial V_1}{\partial y} &= \frac{\partial v}{\partial y}(x, -y)
\end{aligned}$$

Applying the C-R equations for $f(z)$ we see they are satisfied by $F_1(z)$:

$$\begin{aligned}
\frac{\partial U_1}{\partial x}(x, y) &= \frac{\partial u}{\partial x}(x, -y) = \frac{\partial v}{\partial y}(x, -y) = \frac{\partial V_1}{\partial y}(x, y) \\
\frac{\partial U_1}{\partial y}(x, y) &= -\frac{\partial u}{\partial y}(x, -y) = \frac{\partial v}{\partial x}(x, -y) = -\frac{\partial V_1}{\partial x}(x, y)
\end{aligned}$$

Thus, $F_1(z)$ is analytic.

**(b)** This part is similar except we'll find that the C-R equations are not satisfied

$$F_2(z) = f(\overline{z}) = u(x, -y) + iv(x, -y).$$

We can write $F_2(z) = U_2(x, y) + iV_2(x, y)$, where $U_2(x, y) = u(x, -y)$ and $V_2(x, y) = v(x, -y)$. Taking partial derivatives we get

$$\begin{aligned}
\frac{\partial U_2}{\partial x}(x, y) &= \frac{\partial u}{\partial x}(x, -y), & \frac{\partial U_2}{\partial y} &= -\frac{\partial u}{\partial y}(x, -y) \\
\frac{\partial V_2}{\partial x}(x, y) &= \frac{\partial v}{\partial x}(x, -y), & \frac{\partial V_2}{\partial y} &= -\frac{\partial v}{\partial y}(x, -y)
\end{aligned}$$

We see that

$$\frac{\partial U_2}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,-y) = \frac{\partial v}{\partial y}(x,-y) = -\frac{\partial V_2}{\partial y}(x,y)$$
$$\frac{\partial U_2}{\partial y}(x,y) = -\frac{\partial u}{\partial y}(x,-y) = \frac{\partial v}{\partial x}(x,-y) = \frac{\partial V_2}{\partial x}(x,y)$$

Thus, the C-R equations are not satisfied unless all the partials are 0, in which case $f(z)$ is constant.

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**Problem 5.** (10 points)
*Let $f(z) = |z|^2$. Show the $\dfrac{df}{dz}$ exists at $z = 0$, but nowhere else.*

We'll use the definition of the derivative as a limit.

$$f'(z_0) = \lim_{z \to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0}.$$

For $z_0 = 0$ this becomes

$$f'(0) = \lim_{z \to 0} \frac{|z|^2}{z} = \lim_{z \to 0} \frac{z\overline{z}}{z} = \lim_{z \to 0} \overline{z} = 0.$$

Since the limit exists, $f$ is analytic at 0 and $f'(0) = 0$.

For $z \neq 0$ we show the limit does not exist by approaching $z$ from two directions and seeing that we get different limits. Let $z = x + iy$.

Approaching $z$ along a horizontal line we have $\Delta z = \Delta x$ and

$$\lim_{\Delta x \to 0} \frac{|z + \Delta x|^2 - |z|^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + y^2 - (x^2 + y^2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x$$

Approaching $z$ along a vertical line we have $\Delta z = i\Delta y$ and

$$\lim_{\Delta y \to 0} \frac{|z + i\Delta y|^2 - |z|^2}{i\Delta y} = \lim_{\Delta y \to 0} \frac{x^2 + (y + \Delta y)^2 - (x^2 + y^2)}{i\Delta y} = \lim_{\Delta y \to 0} \frac{2y\Delta y + (\Delta y)^2}{i\Delta y} = -2iy.$$

Since $x$ and $y$ are both real, these two limits cannot be equal unless $x = y = 0$. Thus, $f(z)$ is not analytic for $z \neq 0$.

Note: we could also have used the C-R equations on $f(z) = u(x,y) + iv(x,y)$, where $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$.

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**Problem 6.** (10 points)
*Using the principal branch of* log *give a region where $\sqrt{z^2 - 1}$ is analytic.*

Even using the principal branch of log there are several possible answers to this question.

**Answer 1.** The principal branch of $\log(w)$ is defined on $\mathbf{C} - \{\text{negative real axis}\}$. So we need to exclude those $z$ that put $w = z^2 - 1$ on the negative real axis. That is, we need to exclude (make a branch cut on) the imaginary axis and the real interval $[-1, 1]$. This is shown in Figure 1.

**Answer 2.** We write $\sqrt{z^2 - 1} = z\sqrt{1 - 1/z^2}$. Now we need to exclude those $z$ that put $w = 1 - 1/z^2$ on the negative real axis. That is, our branch cut is the real interval $[-1, 1]$. This is shown in Figure 2.

**Answer 3.** We write $\sqrt{z^2 - 1} = \sqrt{z + 1}\sqrt{z - 1}$. Now we need to exclude those $z$ that put either $w = z + 1$ or $w = z - 1$ on the negative real axis. That is, our branch cut is the real interval $(-\infty, 1]$. This is shown in Figure 3.

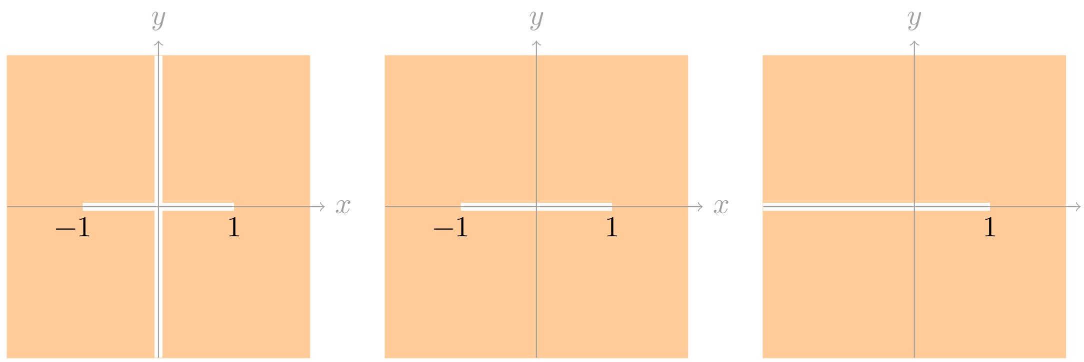


Figure 1. Figure 2. Figure 3.

**Note.** It turns out that in Answer 3 we excluded more than we needed to. This is because we made two branch cuts: $(-\infty, -1]$ for $\sqrt{z + 1}$ and $(-\infty, 1)$ for $\sqrt{z - 1}$. Thus the interval $(-\infty, -1]$ is covered twice and $(-1, 1]$ just once. The square root function changes sign as $z$ crosses from one side of a branch cut to the other. Thus each factor in the product $\sqrt{z + 1}\sqrt{z - 1}$ changes sign as we cross $(-\infty, -1)$. This means the product doesn't change sign and we don't need that portion of the branch cuts!