18.04 Problem Set 3, Spring 2018

Calendar
T Feb. 20: Finish topic 2 notes
W Feb. 21: Reading: Review of 18.02
R Feb. 22: Recitation
F Feb. 23: Reading: Topic 3 notes

Coming next

Feb. 26-Mar. 2: Cauchy's theorem, Cauchy's integral formula

Problem 1. (30: 10,10,10 points)
(a) Compute $\displaystyle\int_C \frac{1}{z}\, dz$, where C is the unit circle around the point $z = 2$ traversed in the counterclockwise direction.

(b) Show that $\int_C z^2\, dz = 0$ for any simple closed curve C in 2 ways.

(i) Apply the fundamental theorem of complex line integrals

(ii) Write out both the real and imaginary parts of the integral as 18.02 integrals of the form $\int_C M\, dx + N\, dy$ and apply Green's theorem to each part.

(c) Consider the integral $\displaystyle\int_C \frac{1}{z}\, dz$, where C is the unit circle. Write out both the real and imaginary parts as 18.02 integrals, i.e. of the form $\int_C M(x,y)\, dx + N(x,y)\, dy$.

Problem 2. (20: 10,10 points)
(a) Let C be the unit circle traversed counterclockwise. Directly from the definition of complex line integrals compute $\displaystyle\int_C \overline{z} dz$.

Is this the same as $\displaystyle\int_C z\, dz$?

(b) Compute $\displaystyle\int_C \overline{z}^2\, dz$ for each of the following paths from 0 to $1+i$.

(i) The straight line connecting the two points.

(ii) The path consisting of the line from 0 to 1 followed by the line from 1 to $1+i$.

Problem 3. (20: 10,10 points)
Let C be the circle of radius 1 centered at $z = -4$. Let $f(z) = 1/(z+4)$. and consider the line integral

$$I = \int_C f(z)\, dz.$$

(a) Does Cauchy's Theorem imply that $I = 0$? Why or why not?

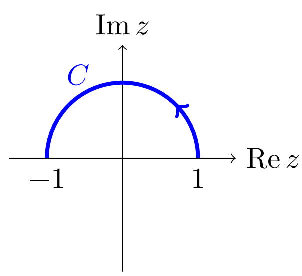
(b) Parametrize the curve C and carry out the calculation to find the value of I. Check that the answer confirms your excellent reasoning in part (a).

Problem 4. (10 points)
Let C be a path from the point $z_1 = 0$ to the point $z_2 = 1 + i$. Find

$$I = \int_C z^9 + \cos(z) - \mathrm{e}^z \, dz$$

in the form $I = a + ib$. Justify your steps.

Problem 5. (15: 10,5 points)
(a) Compute $\int_C z^{1/3} \, dz$, where C the unit semicircle shown. Use the principal branch of $\arg(z)$ to compute the cube root.



(b) Repeat using the branch with $\pi \le \arg(z) < 3\pi$.

Problem 6. (10 points)
Use the fundamental theorem for complex line integrals to show that $f(z) = 1/z$ cannot possibly have an antiderivative defined on $\mathbf{C} - \{0\}$.

Problem 7. (10 points)
Does $\mathrm{Re}\left(\int_C f(z)\, dz\right) = \int_C \mathrm{Re}(f(z))\, dz$? If so prove it, if not give a counterexample.

Problem 8. (10 points)
Are the following simply connected?
(i) The punctured plane.
(ii) The cut plane: $\mathbf{C} - \{\text{nonnegative real axis}\}$.
(iii) The part of the plane inside a circle.
(iv) The part of the plane outside a circle.