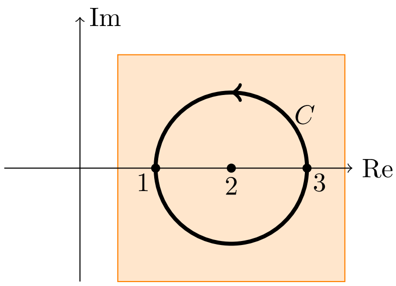
18.04 Problem Set 3, Spring 2018 Solutions

Problem 1. (30: 10,10,10 points)

(a) *Compute* $\displaystyle\int_C \frac{1}{z}\,dz$*, where* C *is the unit circle around the point* $z = 2$ *traversed in the counterclockwise direction.*

Cauchy's theorem says that inside a simply connected region the integral of an analytic function over a closed curve is 0. Since $1/z$ is clearly analytic in a simply connected region containing C and its interior (see figure), Cauchy's theorem applies.

Solution: $\boxed{\displaystyle\int_C \frac{1}{z}\,dz = 0.}$



(b) *Show that* $\int_C z^2\,dz = 0$ *for any simple closed curve* C *in 2 ways.*

(i) Apply the fundamental theorem of complex line integrals

(ii) Write out both the real and imaginary parts of the integral as 18.02 integrals of the form $\int_C M\,dx + N\,dy$ *and apply Green's theorem to each part.*

(i) Let $f(z) = z^2$. We know this has the antiderivative $F(z) = z^3/3$. There fore $\int_C f(z)\,dz = F(z_1) - F(z_0)$, where z_0 and z_1 are the endpoints of the curve C. Since these points coincide, the integral must be 0.

(ii) This requires some algebraic manipulation. Let $z = x + iy$. Then

$$z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$$

So,

$$z^2\,dz = z^2\,(dx + i\,dy) = \left((x^2 - y^2)\,dx - 2xy\,dy\right) + i\left(2xy\,dx + (x^2 - y^2)\,dy\right).$$

It is clear that everything here is defined and differentiable on all of $\mathbf{R}^2$. So we can apply Green's theorem to each part of the integral. (Here R is the interior of the simple closed curve C.)

Real part: $\displaystyle\int_C (x^2 - y^2)\,dx - 2xy\,dy$. So, $M = x^2 - y^2$, $N = -2xy$. Taking the curl we get $N_x - M_y = -2y - (-2y) = 0$. Therefore, by Green's theorem the integral is 0.

Imaginaray part: We have $M = 2xy$, $N = x^2 + y^2$. So, $N_x - M_y = 2x - 2x = 0$. Again, Green's theorem implies the integral is 0.

(c) *Consider the integral* $\displaystyle\int_C \frac{1}{z}\,dz$*, where* C *is the unit circle. Write out both the real and imaginary parts as 18.02 integrals, i.e. of the form* $\int_C M(x,y)\,dx + N(x,y)\,dy$*.*

We have $\dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x-iy}{x^2+y^2}$, So

$$\int_C \frac{1}{z}\,dz = \int_C \frac{x-iy}{x^2+y^2}(dx+i\,dy) = \boxed{\int_c \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy + i\int_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy}$$

Note that the real part uses the radial vector field and the imaginary part uses the tangential (to the circle) vector field.

Problem 2. (20: 10,10 points)
(a) *Let C be the unit circle traversed counterclockwise. Directly from the definition of complex line integrals compute $\int_C \overline{z}dz$.*

Is this the same as $\int_C z\,dz$?

Parametrize C: $\gamma(t) = e^{it}$, with $0 \le t \le 2\pi$. So, $\gamma'(t) = ie^{it}$. Putting this into the integral gives

$$\int_C \overline{z}\,dz = \int_0^{2\pi} \overline{e^{it}}\,i\,e^{it}\,dt = \int_0^{2\pi} i\,dt = \boxed{2\pi i.}$$

This is not the same as $\int_C z\,dz$ which equals 0.

(b) *Compute $\int_C \overline{z}^2\,dz$ for each of the following paths from 0 to $1+i$.*

(i) The straight line connecting the two points.

(ii) The path consisting of the line from 0 to 1 followed by the line from 1 to $1+i$.

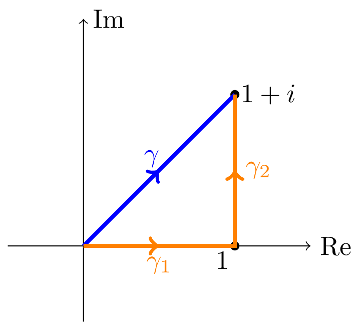
(i) Parametrize the path: $\gamma(t) = t(1+i)$, with $0 \le t \le 1$. So, $\gamma'(t) = 1+i$.
For $z = \gamma(t)$ we have $\overline{z}^2 = t^2(1-i)^2 = -2i\,t^2$. Putting all this into the intgeral gives

$$\int_\gamma \overline{z}^2\,dz = \int_0^1 t^2(-2i)(1+i)\,dt = \frac{2}{3}(1-i).$$

(ii) We have two curves: $\gamma_1(t) = t$, for $0 \le t \le 1$. So, $\gamma_1'(t) = 1$. On γ_1, $\overline{z}^2 = t^2$.
$\gamma_2(t) = 1+it$, for $0 \le t \le 1$. So, $\gamma_2'(t) = i$. On γ_2, $\overline{z}^2 = (1-it)^2 = (1-t^2)-2it$. Integrating each piece separately gives

$$\int_{\gamma_1} \overline{z}^2\,dz = \int_0^1 t^2\,dt = \frac{1}{3}.$$
$$\int_{\gamma_2} \overline{z}^2\,dz = \int_0^1 ((1-t^2)-2it)(i\,dt) = 1 + \frac{2}{3}i.$$

So the integral over the entire path is $\displaystyle\int_{\gamma_1+\gamma_2} \overline{z}^2\,dz = \frac{1}{3} + 1 + \frac{2}{3}i = \boxed{\frac{4}{3} + \frac{2}{3}i.}$



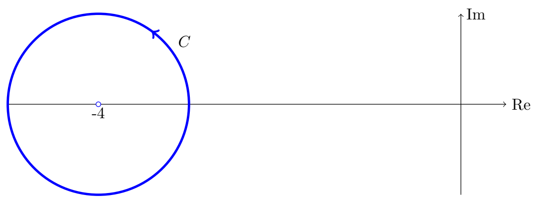
Paths for 2b(i) and 2b(ii).

Problem 3. (20: 10,10 points)
Let C be the circle of radius 1 centered at $z=-4$. Let $f(z)=1/(z+4)$. and consider the line integral

$$I=\int_C f(z)\,dz.$$

(a) *Does Cauchy's Theorem imply that $I=0$? Why or why not?*

No. $f(z)=1/(z+4)$ is not analytic at $z=-4$. Since this is in the interior of C the function is not analytic on a simply connected region containing C, so Cauchy's theorem does not apply.



The singularity $z=-4$ is inside the curve C.

(b) *Parametrize the curve C and carry out the calculation to find the value of I. Check that the answer confirms your excellent reasoning in part (a).*

Parametrize C: $\gamma(t)=-4+e^{it}$, with $0\le t\le 2\pi$. So, $\gamma'(t)=ie^{it}$. Now compute the integral:

$$\int_\gamma \frac{1}{z+4}\,dz=\int_0^{2\pi}\frac{1}{e^{it}}\,ie^{it}\,dt=\int_0^{2\pi} i\,dt=\boxed{2\pi i.}$$

Problem 4. (10 points)
Let C be a path from the point $z_1=0$ to the point $z_2=1+i$. Find

$$I=\int_C z^9+\cos(z)-e^z\,dz$$

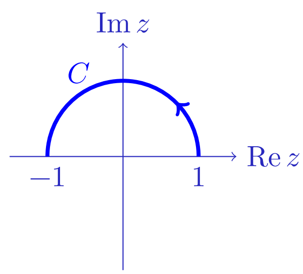
in the form $I=a+ib$. Justify your steps.

We'll apply the fundamental theorem. First, we know that $f(z)=z^9+\cos(z)-e^z$ has antiderivative $F(z) = \frac{z^{10}}{10} + \sin(z) - \mathrm{e}^z$. So, by the fundamental theorem

$$\int_\gamma f(z)\,dz = F(1+i) - F(0) = \frac{(1+i)^{10}}{10} + \sin(1+i) - \mathrm{e}^{1+i}.$$

There is not much purpose in trying to simplify this expression.

Problem 5. (15: 10,5 points)
(a) *Compute $\int_C z^{1/3}\,dz$, where C the unit semicircle shown. Use the principal branch of $\arg(z)$ to compute the cube root.*



We parametrize the curve being sure to put the polar angle in the proper branch of arg: $\gamma(t) = \mathrm{e}^{it}$, where $0 \le t \le \pi$. So,

$$\int_\gamma z^{1/3}\,dz = \int_0^\pi \mathrm{e}^{it/3} i\mathrm{e}^{it} = i\int_0^\pi \mathrm{e}^{i4t/3}\,dt = \frac{3}{4}\mathrm{e}^{i4t/3}\Big|_0^\pi = \frac{3}{4}\left(\mathrm{e}^{i4\pi/3} - 1\right) = \boxed{\frac{3}{8}(-3 - i\sqrt{3}).}$$

(b) *Repeat using the branch with $\pi \le \arg(z) < 3\pi$.*

This is the same as part (a) except we need to use $2\pi \le t \le 3\pi$. So,

$$\int_\gamma z^{1/3}\,dz = \int_{2\pi}^{3\pi} \mathrm{e}^{it/3} i\mathrm{e}^{it} = \frac{3}{4}\mathrm{e}^{i4t/3}\Big|_{2\pi}^{3\pi} = \frac{3}{4}\left(1 - \mathrm{e}^{i8\pi/3}\right) = \frac{3}{4}\left(1 - \mathrm{e}^{i2\pi/3}\right) = \mathrm{e}^{2\pi/3} \cdot \text{part (a)}.$$

Problem 6. (10 points)
Use the fundamental theorem for complex line integrals to show that $f(z) = 1/z$ cannot possibly have an antiderivative defined on $\mathbf{C} - \{0\}$.

We know that over the unit cirle $\int_C \frac{1}{z}\,dz = 2\pi i$. If $1/z$ had an antiderivative on the punctured plane then the fundamental theorem would imply this line integral should be 0. Since it's not, there is no such antiderivative.

Problem 7. (10 points)
Does $\mathrm{Re}\left(\int_C f(z)\,dz\right) = \int_C \mathrm{Re}(f(z))\,dz$? If so prove it, if not give a counterexample.

No! There are many counterexamples. Here's one with the function z over the path $\gamma(t) = t(1+i)$, with $0 \le t \le 1$. Using the fundamental theorem we have

$$\int_\gamma z\,dz = \frac{z^2}{2}\Big|_0^{1+i} = \frac{(1+i)^2}{2} = i.$$

The real part of this is 0.

On the other hand, $\mathrm{Re}(z) = x$. This does not have an antiderivative, so we can't use the fundamental theorem. Therefore we compute it directly:

$$\int_\gamma x\,dz = \int_0^1 t(1+i)\,dt = \frac{1}{2} + i\frac{1}{2}.$$

The real part of this is 1/2, which is not the same as the previous real part.

Problem 8. (10 points)
Are the following simply connected?
(i) The punctured plane.
(ii) The cut plane: $\mathbf{C} - \{\textit{nonnegative real axis}\}$.
(iii) The part of the plane inside a circle.
(iv) The part of the plane outside a circle.

(i) No, not simply connected. There is a hole at the origin. Equivalently, a loop around 0 cannot be contracted to a point.

(ii) Yes, simply connected. Since the cut goes to infinity there are no wholes. Equivalently, every curve in the cut plane can be contracted to a point.

(iii) Yes, a disk is simply connected.

(iv) No, there is a big hole left in the plane by the disk that's been removed.
