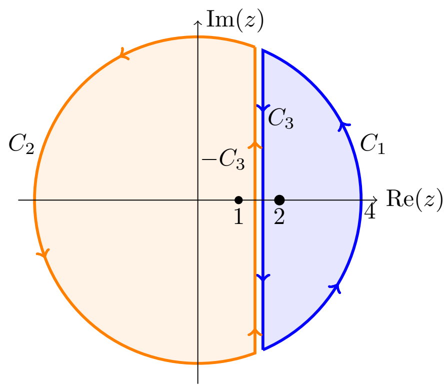
18.04 Problem Set 4, Spring 2018 Solutions

Problem 1. (20: 5,5,5,5 points)
(a) *Use Cauchy's integral formula to compute*

$$\int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}\,dz,$$

where C is the circle of radius 4: $|z|=4$.

Let $f(z)=\dfrac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}$. There are two singularities of f inside the contour C. Let's use the trick of splitting C into two closed contours, each of which surrounds exactly on singularity.



Split $|z|=4$ into two contours.

We have

$$\int_C f(z)\,dz=\int_{C_1+C_3-C_3+C_2} f(z)\,dz=\int_{C_1+C_3} f(z)\,dz+\int_{-C_3+C_2} f(z)\,dz.$$

We compute each integral separately.

C_1+C_3: This contour contains the singularity $z=2$. We let $g(z)=\dfrac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)}$, so $f(z)=g(z)/(z-2)$. Now, $g(z)$ is analytic on and inside C_1+C_3 so, by Cauchy's integral formula we have

$$\int_{C_1+C_3} f(z)\,dz=\int_{C_1+C_3}\frac{g(z)}{z-2}\,dz=2\pi i g(2)=2\pi i.$$

$-C_3+C_2$: This contour contains the singularity $z=1$. We let $g(z)=\dfrac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)}$, so $f(z)=g(z)/(z-2)$. Now, $g(z)$ is analytic on and inside $-C_3+C_2$ so, by Cauchy's integral formula we have

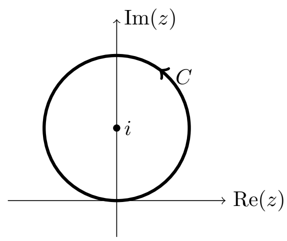
$$\int_{-C_3+C_2} f(z)\,dz=\int_{-C_3+C_2}\frac{g(z)}{z-1}\,dz=2\pi i g(1)=2\pi i.$$

Add the results together we get: the integral in the question equals $4\pi i$.

Note. We could have done this with partial fractions. In essence, that is what we've done. The method shown will be formalized and made easier once we've learned the residue theorem.

(b) *Compute* $\int_C \frac{z^2}{z^2+1}\,dz$*, where* C *is the circle of radius 1 centered at* $z = i$*.*

Let $f(z) = \frac{z^2}{z^2+1} = \frac{z^2}{(z+i)(z-i)}$. The only singularity of f inside C is at $z = -i$.



We let $g(z) = \frac{z^2}{z+i}$, so $f(z) = g(z)/(z-i)$. By Cauchy's integral formula we have

$$\int_C f(z)\,dz = \int_C \frac{g(z)}{z-i}\,dz = 2\pi i g(i) = \boxed{-\pi}.$$

(c) *Let* C *be the circle of radius 2:* $|z| = 2$*. Use Cauchy's integral formula to compute*

$$\int_C \frac{\overline{z}}{z^2-1}\,dz.$$

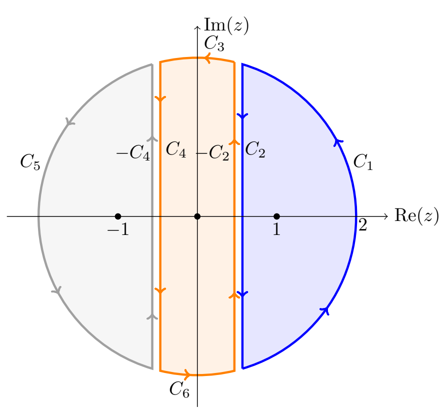
Be careful: $\overline{z}$ *is not analytic, but there is a way around this.*

The trick here is that **on the curve** C we have $\overline{z} = 4/z$. Therefore,

$$\int_C \frac{\overline{z}}{z^2-1}\,dz = \int_C \frac{4}{z(z^2-1)}\,dz.$$

(If you don't believe this you should parametrize C and write out both integrals explicitly. You will see they are the same. Of course, this is special to the circle C. It woudn't be true on every curve.)

Now, $f(z) = \frac{4}{z(z^2-1)}$ has three singularities inside C: at $z = 0, \pm 1$. We use our trick of dividing C. This time into three loops.



So, $\displaystyle\int_C f(z)\,dz = \int_{C_1+C_2-C_2+C_3+C_4-C_4+C_5+C_6} f(z)\,dz = \int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{L_3} f(z)\,dz.$
Here L_1, L_2 and L_3 are the names of the loops as defined below. We now do the same thing as in part (a). We will do it without many comments.

First note: $f(z) = \dfrac{4}{z(z-1)(z+1)}$.

On $L_1 = C_1 + C_2$: Inside this loop f has a singularity at $z = 1$. Let $g(z) = \frac{4}{z(z+1)}$, so $f(z) = g(z)/(z-1)$.

$$\int_{L_1} f(z)\,dz = \int_{L_1} \frac{g(z)}{z-1} = 2\pi i g(1) = 4\pi i.$$

On $L_2 = -C_2 + C_3 + C_4 + C_6$: Inside this loop f has a singularity at $z = 0$. Let $g(z) = \frac{4}{(z+1)(z-1)}$, so $f(z) = g(z)/z$.

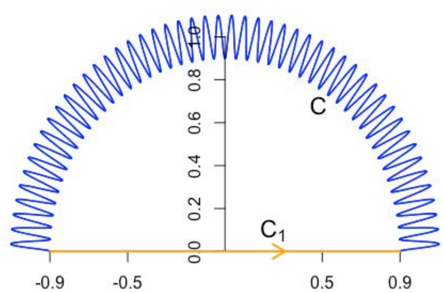
$$\int_{L_2} f(z)\,dz = \int_{L_2} \frac{g(z)}{z} = 2\pi i g(0) = -8\pi i.$$

On $L_3 = -C_4 + C_5$: Inside this loop f has a singularity at $z = -1$. Let $g(z) = \frac{4}{z(z-1)}$, so $f(z) = g(z)/(z+1)$.

$$\int_{L_3} f(z)\,dz = \int_{L_3} \frac{g(z)}{z+1} = 2\pi i g(-1) = 4\pi i.$$

Totaling the contribution from each loop we have our answer: $4\pi i - 8\pi i + 4\pi i = \boxed{0}$.

(d) *Let $\theta = \arg(z)$ Take C to be the wavy contour in the z-plane described by $0 \le \arg(z) \le \pi$; $|z| = 1 - 0.1\cos(100\theta)$. Compute the integral $\displaystyle\int_C z^2\,dz$.*



Method 1. Use the antiderivative:

$$\int_C z^2\,dz = \left.\frac{z^3}{3}\right|_{0.9}^{-0.9} = \boxed{-2(0.9)^3.}$$

Method 2. C is not closed so we close it off by adding the straight line C_1 from -1 to 1 along the x-axis.

Now, by Cauchy's theorem $\displaystyle\int_{C+C_1} z^2\,dz = 0$, so

$$\int_C z^2\,dz = -\int_{C_1} z^2\,dz = -\int_{-0.9}^{0.9} x^2\,dx = \boxed{-2(0.9)^3.}$$

Problem 2. (15: 10,5 points)
(a) *Let $f(z) = z^n$, where n is a positive integer. By directly computing the integral, show that Cauchy's integral formula holds for $f(z_0)$ and Cauchy's formula for derivatives holds for $f'(z_0)$.*

You may need the binomial formula for expanding $(a+b)^n$. As a hint: you may want to make a short argument, based on Cauchy's theorem, reducing the integrals to circles centered on the point of interest.

First $f(z_0)$: Fix z_0 as our point of interest. Suppose C is a simple closed curve going (counterclockwise) around z_0. Our goal is to show that

$$\frac{1}{2\pi i}\int_C \frac{z^n}{z-z_0} = z_0^n.$$

By the extended version of Cauchy's theorem we know

$$\int_C \frac{z^n}{z-z_0} = \int_{C_r} \frac{z^n}{z-z_0},$$

where C_r is a small circle centered at z_0 and entirely inside C. This last integral we can compute directly.

Parametrize C_r: $\gamma(t) = z_0 + r\mathrm{e}^{i\theta}$, with $0 \le \theta \le 2\pi$. $\gamma'(\theta) = ir\mathrm{e}^{i\theta}$. So,

$$\frac{1}{2\pi i}\int_{C_r} \frac{z^n}{z-z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(z_0+r\mathrm{e}^{i\theta})^n}{r\mathrm{e}^{i\theta}}\,ir\mathrm{e}^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} (z_0+r\mathrm{e}^{i\theta})^n\,d\theta$$

The binomial formula tells us that

$$\frac{1}{2\pi}\int_0^{2\pi} (z_0+r\mathrm{e}^{i\theta})^n\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} (z_0^n + nz_0^{n-1}r\mathrm{e}^{i\theta} + a_2 z_0^{n-2}r^2\mathrm{e}^{i2\theta} + \ldots + r^n\mathrm{e}^{in\theta})\,d\theta,$$

where the a_j are binomial coefficients whose exact value doesn't concern us. Since $\int_0^{2\pi} \mathrm{e}^{im\theta}\,d\theta = 0$ for $m = 1, 2, 3, \ldots$, after integration, the only nonzero term in the integral is

$$\frac{1}{2\pi}\int_0^{2\pi} z_0^n\,d\theta = z_0^n. \qquad \text{QED}$$

Second $f'(z_0)$. This is similar. We have to show that

$$\frac{1}{2\pi i}\int_C \frac{z^n}{(z-z_0)^2} = nz_0^{n-1}.$$

Using the binomial theorem we get

$$\begin{aligned}
\frac{1}{2\pi i}\int_{C_r}\frac{z^n}{(z-z_0)^2}\,dz &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{(z_0+r\mathrm{e}^{i\theta})^n}{r^2\mathrm{e}^{i2\theta}}\,ir\mathrm{e}^{i\theta}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{(z_0^n+nz_0^{n-1}r\mathrm{e}^{i\theta}+a_2z_0^{n-2}r^2\mathrm{e}^{i2\theta}+\ldots+r^n\mathrm{e}^{in\theta})}{r\mathrm{e}^{i\theta}}\,d\theta, \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{(z_0^n\mathrm{e}^{-i\theta}+nz_0^{n-1}r+a_2z_0^{n-2}r^2\mathrm{e}^{i\theta}+\ldots+r^n\mathrm{e}^{i(n-1)\theta})}{r}\,d\theta
\end{aligned}$$

After integration, the only nonzero term is

$$\frac{1}{2\pi}\int_0^{2\pi} nz_0^{n-1}\,d\theta = nz_0^{n-1}. \quad \text{QED}$$

(b) *Let $P(z)=c_o+c_1z+c_2z^2+c_3z^3$. Let C be the circle $|z|=a$, for $a>0$. Compute the integral*

$$\int_C P(z)z^{-n}dz \ \textit{ for } n = \textit{0, 1, 2, ...}$$

This is harder to express that to see. The key is our well known fact:

$$\int_C z^m\,dz = \begin{cases} 0 & \text{if } m\neq -1 \\ 2\pi i & \text{if } m=-1 \end{cases}$$

So

$$\int_C P(z)z^{-n}\,dz = \int_C c_0z^{-n}+c_1z^{1-n}+c_3z^{3-n}+c_3z^{3-n}\,dz$$

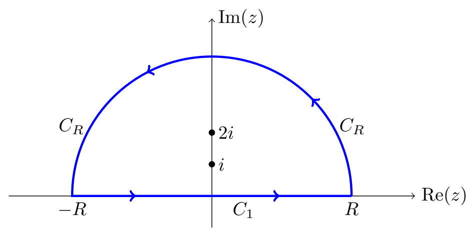
Most of these terms integrate to 0. We have:
If $n=1$ then the integral is $2\pi ic_0$.
If $n=2$ then the integral is $2\pi ic_1$.
If $n=3$ then the integral is $2\pi ic_2$.
If $n=4$ then the integral is $2\pi ic_3$.
For all other n the integral is 0.

Problem 3. (15: 5,5,5 points)
(a) *Compute* $\displaystyle\int_C \frac{|z|\mathrm{e}^z}{z^2}\,dz$ *where C is the circle $|z|=2$.*

Since $|z|=2$ on C, the line integral $\displaystyle\int_C\frac{|z|\mathrm{e}^z}{z^2}\,dz=\int_C\frac{2\mathrm{e}^z}{z^2}\,dz$. We can compute this last integral using Cauchy's integral formula for the derivative.
Let $g(z)=2\mathrm{e}^z$. Then $\int_C\frac{g(z)}{z^2}\,dz=(2\pi i)g'(0)=\boxed{4\pi i}$.

(b) *Compute* $\displaystyle\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(x^2+4)}\,dx$.

Hint: integrate over the closed path shown below. Show that as R goes to infinity the contribution of the integral over C_R becomes 0.



There is an example like this in the topic 4 notes. The strategy is:

1. Let $f(z) = \dfrac{f(z)}{(z^2+1)(z^2+4)}$ and use Cauchy's formula to evaluate $\displaystyle\int_{C_1+C_R} f(z)\,dz$.

2. Notice that $\displaystyle\lim_{R\to\infty}\int_{C_1} f(z)\,ds =$ the integral we want to compute.

3. Show that $\displaystyle\lim_{R\to\infty}\int_{C_R} f(z)\,ds = 0$.

Step 1. $f(z) = \dfrac{1}{(z+i)(z-i)(z+2i)(z-2i)}$. For R large enough the singularities of f inside $C_1 + C_R$ are $z = i$ and $z = 2i$. As before, we break $C_1 + C_R$ into two loops, each enclosing one of the singularities. We won't bother drawing or naming the loops.

Loop around $z = i$: Let $g(z) = \dfrac{1}{(z+i)(z+2i)(z-2i)}$. So, by Cauchy,

$$\int_{\text{loop}} f(z)\,dz = \int_{\text{loop}} \frac{g(z)}{z-i}\,dz = 2\pi i g(i) = \frac{\pi}{3}.$$

Loop around $z = 2i$: Let $g(z) = \dfrac{1}{(z+i)(z-i)(z+2i)}$. So, by Cauchy,

$$\int_{\text{loop}} f(z)\,dz = \int_{\text{loop}} \frac{g(z)}{z-2i}\,dz = 2\pi i g(2i) = -\frac{\pi}{6}.$$

Totaling, $\displaystyle\int_{C_1+C_R} f(z)\,dz = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$.

Step 2. This is clear: parametrize C_1 as $\gamma(x) = x$, with $-R \le x \le R$; $\gamma'(x) = 1$. So,

$$\int_{C_1} f(z)\,dz = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+4)}\,dx$$

this clearly goes to the integral from $-\infty$ to ∞ in the limit.

Note: we should verify that the improper integral is absolutely convergent. But this is clear since the denominator grows like x^4 and is never 0 on the real axis.

Step 3. Parametrize C_R: $\gamma(\theta) = Re^{i\theta}$, with $0 \le \theta \le \pi$. So,

$$\begin{aligned}
\left| \int_{C_R} f(z)\, dz \right| &= \left| \int_0^\pi \frac{1}{(R^2 e^{i2\theta}+1)(R^2 e^{i2\theta+4})} iRe^{i\theta}\, d\theta \right| \\
&\le \int_0^\pi \left| \frac{1}{(R^2 e^{i2\theta}+1)(R^2 e^{i2\theta+4})} iRe^{i\theta} \right| d\theta && \text{(triangle inequality)} \\
&= \int_0^\pi \frac{1}{|R^2 e^{i2\theta}+1|\, |R^2 e^{i2\theta+4}|} R, d\theta \\
le &\int_0^\pi \frac{1}{|R^2-1|\,|R^2-4|} R, d\theta && \text{(also by triangle inequality)} \\
&= \frac{R\pi}{|R^2-1|\,|R^2-4|}
\end{aligned}$$

This goes to 0 as R goes to ∞ as required by our strategy.

Putting the steps together:

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\, dx = \lim_{R\to\infty} \int_{C_1+C_R} f(z)\, dz = \lim_{R\to\infty} \frac{\pi}{6} = \boxed{\frac{\pi}{6}.}$$

(c) *Show that* $\displaystyle\int_{|z|=2} \frac{1}{z^2(z-1)^3}\, dz = 0$ *in two different ways.*

(i) Use Cauchy's integral formula. You'll need to divide the contour to isolate each of the singularities of the integrand.

(ii) First, show that the integral doesn't change if you replace the contour by the curve $|z| = R$ for $R > 2$. Next, show that this integral must go to 0 as R goes to infinity.

Both parts of this problem are similar to what we've done in earlier problems. We give brief answers.

(i) Let $f(z) = 1/(z^2(z-1)^3)$. The countour $|z| = 2$ contains singularities of f at $z = 0$ and $z = 1$.

Loop around 0: Let $g(z) = 1/(z-1)^3$.

$$\int_{|z|=2} f(z), dz = \int_{|z|=2} \frac{g(z)}{z^2} = 2\pi i g'(0) = -6\pi i.$$

Loop around 1: Let $g(z) = 1/z^2$.

$$\int_{|z|=2} f(z), dz = \int_{|z|=2} \frac{g(z)}{(z-1)^3} = 2\pi i g''(1) = 6\pi i.$$

Adding these together we get 0.

(ii) Since all the singularities of f are inside $|z| = 2$ the extended Cauchy theorem allow us to replace the contour $|z| = 2$ by $|z| = R$ without changing the integral. Then,

$$\begin{aligned}\left|\int_{|z|=R} f(z)\,dz\right| \le \int_{|z|=R} |f(z)|\,|dz| \\ &= \int_0^{2\pi} \frac{R}{R^2|Re^{i\theta}-1|^3}\,d\theta \\ &\le \int_0^{2\pi} \frac{R}{R^2(R-1)^3}\,d\theta \\ &= \frac{2\pi R}{R^2(R-1)^3}\end{aligned}$$

This clearly goes to 0 as R goes to infinity.

Problem 4. (5 points)
Suppose f is analytic on and inside a simple closed curve C. Assume $f(z) = 0$ for z on C. Show $f(z) = 0$ for all z inside C.

Take an arbitrary point z_0 inside C. Cauchy's integral formula says

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_c 0\,dz = 0. \quad \text{QED}$$

Problem 5. (10 points)
Let γ be a simple closed curve that goes through the point $1+i$. Let

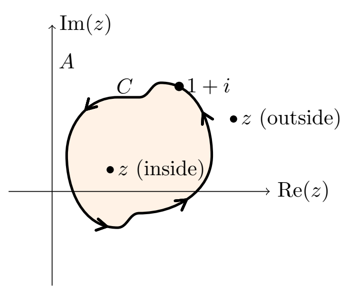
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{\cos(w)}{w-z}\,dw.$$

Find the following limits:
(i) $\lim_{z\to 1+i} f(z)$, where z goes to $1+i$ from outside γ.
(ii) $\lim_{z\to 1+i} f(z)$, where z goes to $1+i$ from inside γ.

(i) If z is outside γ then $\dfrac{\cos(w)}{w-z}$ is analytic on and inside γ. So, Cauchy's theorem says the path integral

$$f(z) = \frac{1}{2\pi i}\int_C \frac{\cos(w)}{w-z}\,dw = 0$$

This is constant, so the limit from the outside as $z \to 1+i$ is 0.



(ii) (i) If z is inside γ then Cauchy's integral formula says

$$f(z) = \frac{1}{2\pi i}\int_C \frac{\cos(w)}{w-z}\,dw = \cos(1+i)$$

This is constant, so the limit from the inside as $z \to 1+i$ is $\cos(1+i)$.

Problem 6. (10: 5,5 points)
(a) *Suppose that $f(z)$ is analytic on a region A that contains the disk $|z - z_0| \le r$. Use Cauchy's integral formula to prove the mean value property*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\mathrm{e}^{i\theta})\,d\theta.$$

This is in the notes: parametrize the curve $C_r : |z - z_0| = r$ by $\gamma(\theta) = z_0 + r e^{i\theta}$, with $0 \le \theta \le 2\pi$. By Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + r\mathrm{e}^{i\theta})}{r\mathrm{e}^{i\theta}\, i r\mathrm{e}^{i\theta}}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\mathrm{e}^{i\theta})\,d\theta. \quad \text{QED}$$

(b) *Prove the more general formula*

$$f^{(n)}(z_0) = \frac{n!}{2\pi r^n}\int_0^{2\pi} f(z_0 + r\mathrm{e}^{i\theta})\mathrm{e}^{-in\theta}\,d\theta.$$

Use the same curve as in part (a). This time use Cauchy's formula for derivatives.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + r\mathrm{e}^{i\theta})}{r^{n+1}\mathrm{e}^{i(n+1)\theta}} i r\mathrm{e}^{i\theta}\,d\theta = \frac{n!}{2\pi r^n}\int_0^{2\pi} f(z_0 + r\mathrm{e}^{i\theta})\mathrm{e}^{-in\theta}\,d\theta.$$

This is what we needed to prove.

Problem 7. (20: 4,4,4,4,4 points)
Let C be the curve $|z| = 2$. Explain why each of the following integrals is 0.

(a) $\displaystyle\int_C \frac{z}{z^2+35}\,dz.$

Solution: The singularities of $\dfrac{z}{z^2+35}$ are at $\pm i\sqrt{35}$. These are both outside the circle $|z| = 2$. So, by Cauchy's theorem the integral is 0.

(b) $\displaystyle\int_C \frac{\cos(z)}{z^2-6z+10}\,dz.$

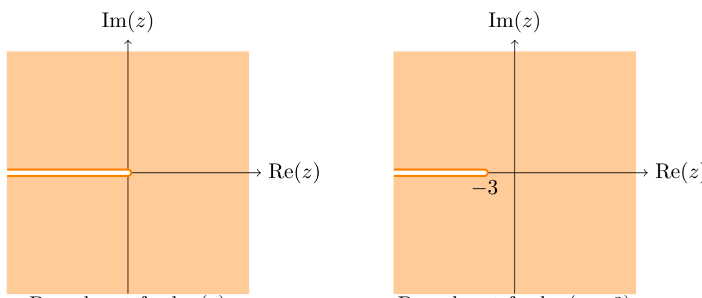
Solution: The singularities of $\dfrac{\cos(z)}{z^2-6z+10}$ are at the roots of $z^2 - 6z + 10$. These are at $3 \pm i$, which are outside the circle $|z| = 2$. Again, by Cauchy's theorem the integral is 0.

(c) $\displaystyle\int_C \mathrm{e}^{-z}(2z+1)\,dz.$

Solution: The integrand is entire. So by Cauchy's theorem the integral is 0.

(d) $\displaystyle\int_C \log(z+3)\,dz$ *(principal branch of* $\log$*).*

Solution: The principal branch of $\log(z)$ has a cut on the negative real axis. So $\log(z+3)$ has its branch cut on $x \le -3$. Since this is outside the circle $|z| = 2$, the integrand is analytic on and inside the curve, so by Cauchy's theorem the integral is 0.



Branch cut for $\log(z)$ Branch cut for $\log(z+3)$

(e) $\displaystyle\int_C \sec(z/2)\,dz.$

Solution: $\sec(z/2) = 1/\cos(z/2)$ has singularities at $\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots$. Since these are all outside the curve $|z| = 2$, Cauchy's theorem implies the integral is 0.

Extra problems not to be scored. If you want someone to look at them, please turn them in separately to Jerry.

Problem 8. (0 points)

Show $\displaystyle\int_0^{\pi} \mathrm{e}^{\cos\theta}\cos(\sin(\theta))\,d\theta = \pi$. *Hint, consider* e^z/z *over the unit circle.*

Solution: (Follow the hint.) Parametrize the unit circle as $\gamma(\theta) = \mathrm{e}^{i\theta}$, with $0 \le \theta \le 2\pi$. So,

$$\begin{aligned}\int_\gamma \frac{\mathrm{e}^z}{z}\,dz &= \int_0^{2\pi} \frac{\mathrm{e}^{cos\theta + i\sin\theta}}{\mathrm{e}^{i\theta}} i\mathrm{e}^{i\theta}\,d\theta = i\int_0^{2\pi} \mathrm{e}^{cos\theta+i\sin}\,d\theta \\ &= i\int_0^{2\pi} \mathrm{e}^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta))\,d\theta = \int_0^{2\pi} \mathrm{e}^{\cos\theta}(i\cos(\sin\theta) - \sin(\sin\theta))\,d\theta.\end{aligned}$$

This is close to what we want. Let's use Cauchy's integral formula to evaluate it and then extract the value we need. By Cauchy the integral is $2\pi i e^0 = 2\pi i$. So,

$$\int_0^{2\pi} \mathrm{e}^{\cos\theta}(i\cos(\sin\theta) - \sin(\sin\theta))\,d\theta = 2\pi i.$$

Taking the imaginary part we have

$$\int_0^{2\pi} \mathrm{e}^{\cos\theta}\cos(\sin\theta)\,d\theta = 2\pi.$$

This integral goes to 2π, while our integral goes to π. By symmetry ours is half the above. (It might be easier to see this if you use the limits $[-\pi, \pi]$ instead of $[0, 2\pi]$.)

Answer: $\boxed{\pi}$ (as we were supposed to show).

Problem 9. (0 points)
(a) *Suppose $f(z)$ is analytic on a simply connected region A and γ is a simple closed curve in A. Fix z_0 in A, but not on γ. Use the Cauchy integral formulas to show that*

$$\int_\gamma \frac{f'(z)}{z - z_0}\, dz = \int_\gamma \frac{f(z)}{(z - z_0)^2}\, dz.$$

Since A is simply connected we know f and f' are analytic on and inside γ. There are two cases: (i) z_0 is inside γ, (ii) z_0 is outside γ.

In case (i) we can use Cauchy's formulas.

$$\int_\gamma \frac{f'(z)}{z - z_0}\, dz = 2\pi i f'(z_0) \qquad \text{(by Cauchy's integral formula.)}$$

$$\int_\gamma \frac{f(z)}{(z - z_0)^2}\, dz = 2\pi i f'(z_0) \qquad \text{(by Cauchy's formula for derivatives.)}$$

These are the same.

In case (ii) both integrands are analytic on and inside γ, so both integrals are 0 by Cauchy's theorem. Again, the integrals are the same. QED

(b) *Challenge: Redo part (a), but drop the assumption that A is simply connected.*

Let $g(z) = \frac{f(z)}{z - z_0}$. g is analytic on a neighborhood of γ. Note: $g'(z) = \dfrac{f'(z)}{z - z_0} - \dfrac{f(z)}{(z - z_0)^2}$.
So,

$$\int_\gamma \frac{f'(z)}{z - z_0}\, dz - \int_\gamma \frac{f(z)}{(z - z_0)^2}\, dz = \int_\gamma g'(z)\, dz = 0.$$

It equals 0 because the integral of a derivative around a closed curve is 0. So, the two integrals on the left side are equal.

Problem 10. (0 points)
Suppose $f(z)$ is entire and $\lim_{z \to \infty} \dfrac{f(z)}{z} = 0$. Show that $f(z)$ is constant.

You may use Morera's theorem: if $g(z)$ is analytic on $A - \{z_0\}$ and continuous on A, then f is analytic on A.

Solution: Let $g(z) = \dfrac{f(z) - f(0)}{z}$. Since $g(z)$ is analytic on $\mathbf{C} - \{0\}$ and continuous on $\mathbf{C}$ it is analytic on all of $\mathbf{C}$, by Morera's theorem

We claim $g(z) \equiv 0$.

Suppose not, then we can pick a point z_0 with $g(z_0) \neq 0$. Since $g(z)$ goes to 0 as $|z|$ gets large we can pick R large enough that $|g(z)| < |g(z_0)|$ for all $|z| = R$. But this violates the maximum modulus theorem, which says that the maximum modulus of $g(z)$ on the disk

$|z| \le R$ occurs on the circle $|z| = R$. This disaster means our assumption that $g(z) \neq 0$ was wrong. We conclude $g(z) \equiv 0$ as claimed.

This means that $f(z) = f(0)$ for all z, i.e. $f(z)$ is constant.

Problem 11. (0 points)

(a) *Compute* $\displaystyle\int_C \frac{\cos(z)}{z}\,dz$*, where* C *is the unit circle.*

Solution: $2\pi i \cos(0) = 2\pi i$.

(b) *Compute* $\displaystyle\int_C \frac{\sin(z)}{z}\,dz$*, where* C *is the unit circle.*

Solution: $2\pi i \sin(0) = 0$.

(c) *Compute* $\displaystyle\int_C \frac{z^2}{z-1}\,dz$*, where* C *is the circle* $|z| = 2$.

Solution: $2\pi i z^2|_{z=1} = 2\pi i$.

(d) *Compute* $\displaystyle\int_C \frac{e^z}{z^2}\,dz$*, where* C *is the circle* $|z| = 1$.

Solution: $2\pi i \, \frac{de^z}{dz}\big|_{z=0} = 2\pi i$

(e) *Compute* $\displaystyle\int_C \frac{z^2-1}{z^2+1}\,dz$*, where* C *is the circle* $|z| = 2$.

Solution: Singularities are at $\pm i$.

$$\text{Integral } = 2\pi i \frac{-2}{2i} + 2\pi i \frac{-2}{-2i} = 0.$$

(f) *Compute* $\displaystyle\int_C \frac{1}{z^2+z+1}\,dz$ *where* C *is the circle* $|z| = 2$.

Solution: There are two roots. Splitting the contour as we've done several times leads to a total integral of 0.