18.04 Problem Set 5, Spring 2018

Calendar
W Mar. 15: Reading: topics 5,6 notes
R Mar. 16: Recitation
F Mar. 17: Reading: start topic 7

Coming next

Mar. 20-Mar. 24: Taylor and Laurent series

Problem 1. (15: 5,10 points)
Let $u(x, y) = x^3 - 3xy^2 + 2x$.

(a) Verify that u is harmonic.

(b) Find a harmonic conjugate v for u in two ways.

(i) Use the Cauchy-Riemann equations to first find v_x and v_y and then find v by integrating these expressions.

(ii) Let $f = u + iv$, then $f' = u_x - iu_y$. Recognize this as a function of z and integrate.

Problem 2. (20: 10,10 points)
(a) Suppose $u(x, y) = x/r^2$, where r is the usual polar r. Show that u is harmonic and find a harmonic conjugate v such that $f = u + iv$ is analytic.

(b) Same question for $u(x, y) = (x^2 - y^2)/r^4$.

Problem 3. (10 points)
Suppose $\mathbf{F} = ((x^2 - y^2)/r^4, 2xy/r^4)$ is a velocity field. Show that $\mathbf{F}$ is divergent free and irrotational (curl free) and find a complex potential function for $\mathbf{F}$.

Hint: Find the complex potential first.

Problem 4. (15: 10,5 points)
Let A be the region bounded by the positive x axis and the ray $x = y$ in the first quadrant.

(a) Find a nonzero function ψ that is harmonic on A and is 0 on the boundary.

Hint: Start with the same question on the upper half-plane $y > 0$.

(b) Let's interpet the level curves of ψ as the streamlines for an incompressible, irrotational flow. Give the velocity field of this flow.

Problem 5. (15: 5,5,5 points)
Consider the complex potential for a fluid given by $\Phi(z) = Az^3$, where $A > 0$:

(a) Find the potential ϕ, the stream-function ψ and the velocity field (u, v).

(b) Sketch the streamlines and the velocity field in the complex plane.

(c) Use this to find an incompressible, irrotational flow in a wedge (for some angle)?

What is the angle of the wedge you can do with this solution?

Problem 6. (10 points)
Suppose the vector field $\mathbf{F} = (u, v)$ is divergence free and irrotational (curl free). Show that u is a harmonic function

Problem 7. (15: 5,5,5 points)
Let A be the unit disk. Assume it is made of a heat conducting material and that in our two dimensional world it only loses heat through its boundary. Then at steady state the temperature $T(x, y)$ in the disk is a harmonic function.

Suppose we hold the temperature of the boundary fixed at

$$T(\mathrm{e}^{i\theta}) = T(\cos(\theta), \sin(\theta)) = \sin^2(\theta).$$

(a) What is the temperature at the center of the disk?

(b) What is the maximum temperature on the disk?

(c) What is the minimum temperature on the disk?

Challenge. Find the temperature $T(x, y)$ throughout the disk.

Problem 8. (10: 5,5 points) (Hints of Taylor and Laurent series.)
Consider the following infinite series.

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots$$

(a) Remember your calculus and use the ratio test to say for what z the series converges.

(b) Let C be a large circle with center at the origin. What is the value of $\displaystyle\int_C f(z)\, dz$?