18.04 Problem Set 5, Spring 2018 Solutions

Problem 1. (15: 5,10 points)
Let $u(x,y) = x^3 - 3xy^2 + 2x$.

(a) *Verify that u is harmonic.*

(b) *Find a harmonic conjugate v for u in two ways.*

(i) Use the Cauchy-Riemann equations to first find v_x and v_y and then find v by integrating these expressions.

(ii) Let $f = u + iv$, then $f' = u_x - iu_y$. Recognize this as a function of z and integrate.

answers: (a) This is a straightforward calculation:

$$u_x = 3x^2 - 3y^2 + 2,\ u_{xx} = 6x, \quad u_y = -6xy,\ u_{yy} = -6x.$$

Therefore $\nabla^2 u = u_{xx} + u_{yy} = 0$. QED

(b) (i) By Cauchy-Riemann $v_x = -u_y = 6xy$, $v_y = u_x = 3x^2 - 3y^2 + 2$ Integrating v_x with respect to x we get $v = 3x^2y + g(y)$, where $g(y)$ is a function of y alone.

Integrating v_y with respect to y we get $v = 3x^2 - y^3 + 2y + h(x)$, where $h(x)$ is a function of x alone.

Comparing the two expressions for v we get $g(y) = -y^3 + 2y + h(x)$. Since g is a function of y alone, we must have $h(x)$ is constant. So, $g(y) = -y^3 + 2y + c$ and

$$\boxed{v = 3x^2y - y^3 + 2y + c,}$$

where c is an arbitrary constant.

(ii)
$$f' = u_x - iu_y = 3x^2 - y^2 + 2 + i6xy = 3(x^2 - y^2 + i2xy) + 2 = 3z^2 + 2.$$

Therefore $f = z^3 + 2z + C = (x^3 - xy^2 + 2x + c_1) + i(3x^2y - y^3 + 2y + c_2)$, where $C = c_1 + ic_2$. So,

$$v = \text{Im}(f) = 3x^2y - y^3 + 2y + c_2.$$

Exactly as in part (i).

Problem 2. (20: 10,10 points)
(a) *Suppose $u(x,y) = x/r^2$, where r is the usual polar r. Show that u is harmonic and find a harmonic conjugate v such that $f = u + iv$ is analytic.*

Solution: Before we start: we'll need the following formulas to help manage the algebra.

$$r = \sqrt{x^2 + y^2}, \quad \frac{\partial r}{\partial x} = r_x = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = r_y = \frac{y}{r} \tag{1}$$

First check that u is harmonic.

$$\begin{aligned} u &= xr^{-2} \\ u_x &= r^{-2} - 2x^2r^{-4} \qquad & u_{xx} &= -2xr^{-4} - 4xr^{-4} + 8x^3r^{-6} = -6xr^{-4} + 8x^3r^{-6} \\ u_y &= -2xyr^{-4} \qquad & u_{yy} &= -2xr^{-4} + 8xy^2r^{-6} \end{aligned}$$

So,

$$u_{xx} + u_{yy} = -8xr^{-4} + 8x^3r^{-6} + 8xy^2r^{-6} = -8xy^{-4} + 8x(x^2+y^2)r^{-6} = 0.$$

This shows u is harmonic. (Note: we don't really need this, since showing that u is the real part of an analytic f will guarantee it is harmonic. On the other hand, knowing it is harmonic means our search for an analytic f is guaranteed to be successful.)

Now, we'll find f directly by integrating $f' = u_x = iu_y$.

$$f' = u_x - iu_y = \frac{-x^2 + y^2 + i2xy}{r^4} = -\frac{(x-iy)^2}{r^4} = -\frac{\overline{z}^2}{z^2\overline{z}^2} = -\frac{1}{z^2}.$$

Thus, $\boxed{f(z) = \frac{1}{z}} = \frac{\overline{z}}{r^2} = \frac{x-iy}{r^2}$ and $\boxed{v = \text{Im}(f) = -\frac{y}{r^2}.}$

(b) *Same question for $u(x,y) = (x^2 - y^2)/r^4$.*

Solution: In this case we won't bother checking that u is harmonic. Once we find the analytic f we'll know that u is harmonic. First we do the algebra to find $f' = u_x - iu_y$. You may have to fill in a few algebraic steps to see how we did the full computation.

$$\begin{aligned}
u &= (x^2 - y^2)r^{-4} \\
u_x &= 2xr^{-4} - 4(x^2-y^2)xr^{-6} = \frac{2xr^2 - 4x^3 + 4xy^2}{r^6} = \frac{-2(x^3 - 3xy^2)}{r^6} \\
u_y &= -2yr^{-4} - 4(x^2-y^2)yr^{-6} = \frac{-2yr^2 - 4x^2y + 4y^3}{r^6} = \frac{-2(3x^2y - y^3)}{r^6}
\end{aligned}$$

So

$$f' = u_x - iu_y = \frac{-2[x^3 - 3xy^2 - i(3x^2y - y^3)]}{r^6} = \frac{-2(x-iy)^3}{r^6} = -2\frac{\overline{z}^3}{z^3\overline{z}^3} = -\frac{2}{z^3}.$$

Thus, $\boxed{f(z) = \frac{1}{z^2}} = \frac{\overline{z}^2}{r^4} = \frac{x^2 - y^2 - i2xy}{r^4}$ and $\boxed{v = \text{Im}(f) = -\frac{2xy}{r^4}.}$

Problem 3. (10 points)
Suppose $\mathbf{F} = ((x^2-y^2)/r^4, 2xy/r^4)$ is a velocity field. Show that $\mathbf{F}$ is divergent free and irrotational (curl free) and find a complex potential function for $\mathbf{F}$.

Hint: Find the complex potential first.

Solution: We follow the hint to find the complex potential Φ. Let $\mathbf{F} = (u,v)$, then

$$\Phi' = u - iv = \frac{x^2 - y^2 - 2ixy}{r^4} = \frac{(x-iy)^2}{r^4} = \frac{1}{z^2}.$$

So, $\boxed{\Phi = -\frac{1}{z}.}$ Since we found a complex potential fo $\mathbf{F}$ we have also shown it is divergence and curl free.

Problem 4. (15: 10,5 points)
Let A be the region bounded by the positive x axis and the ray $x = y$ in the first quadrant.

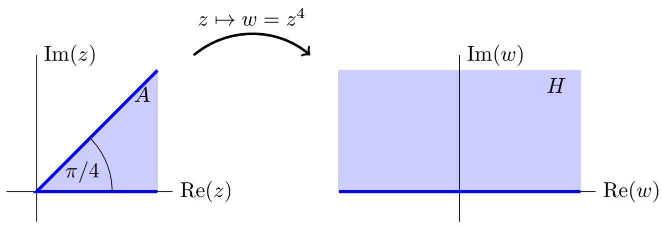
(a) *Find a nonzero function ψ that is harmonic on A and is 0 on the boundary.*

Hint: Start with the same question on the upper half-plane $y > 0$.

(b) *Let's interpet the level curves of ψ as the streamlines for an incompressible, irrotational flow. Give the velocity field of this flow.*

answers: (a) Call the upper half-plane H. The idea is to map A to H, using an analytic function g, then 'pull back' the harmonic function on H to A using composition of functions.

The map $w = z^4$ maps A to H and sends the boundary of A to the boundary of H. (See figure)



$w = z^4$ maps the wedge to the half-plane.

Now we need to find a harmonic function on H that is 0 on the boundary. One easy one is $\psi_H(x, y) = y$. In terms of $w = x + iy$ we have $\psi_H(w) = \text{Im}(w)$.

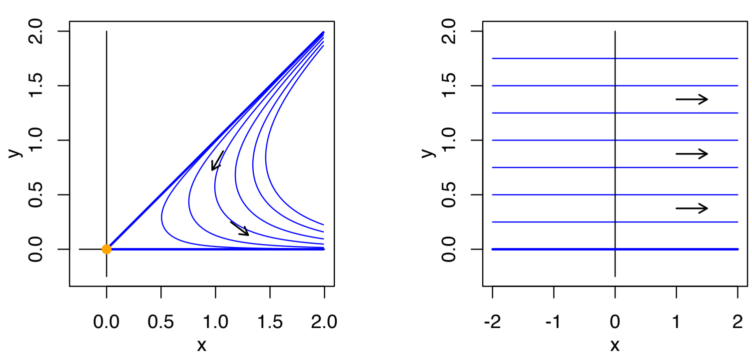
By pulling back the function we mean $\psi(z) = \psi_H(g(z))$. So, if $z = x + iy$,

$$\boxed{\psi(z) = \psi_H(z^4) = \text{Im}(z^4) = 4x^3y - 4xy^3.}$$

(b) Since $\psi = \text{Im}(z^4)$, we get without effort that the complex potential for the flow is $\Phi(z) = z^4$. If we write $\Phi' = u - iv$, then the velocity field is $\mathbf{F} = (u, v)$. Computing:

$$\Phi' = 4z^3 = 4(x^3 - 3xy^2) + 4i(3x^2y - y^3)$$

So, $\boxed{\mathbf{F} = (4(x^3 - 3xy^2, -4(3x^2y - y^3).}$



ψ_H (right figure) is 'pulled back' to the stream function $\psi = \psi_H(g(z))$ (left).

In the figure we only show streamlines in the wedge of interest. The flow associated to $\Phi(z) = z^4$ is defined on the entire plane, which is divide into 8 wedges, each containing streamlines which are wholly contained in the wedge.

Problem 5. (15: 5,5,5 points)
Consider the complex potential for a fluid given by $\Phi(z) = Az^3$, where $A > 0$:

(a) *Find the potential ϕ, the stream-function ψ and the velocity field (u, v).*

(b) *Sketch the streamlines and the velocity field in the complex plane.*

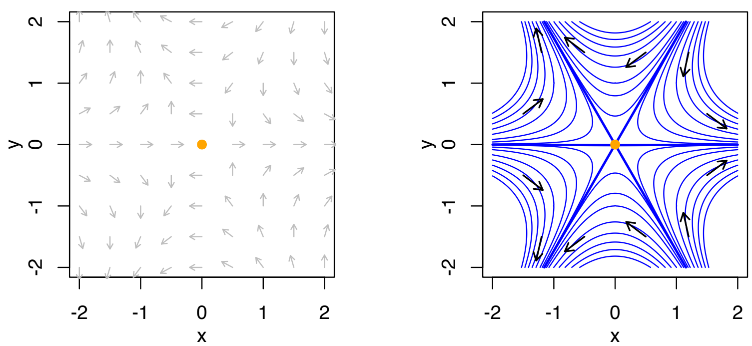
(c) *Use this to find an incompressible, irrotational flow in a wedge (for some angle)? What is the angle of the wedge you can do with this solution?*

answers: This problem is similar to the previous one. There we had the complex potential $\Phi(z) = z^4$. Here the potential is

$$\Phi(z) = Az^3 = A(x+iy)^3 = A(x^3 - 3xy^2) + iA(3x^2y - y^3).$$

(a) So, $\boxed{\phi = \text{Re}(\Phi) = A(x^3 - 3xy^2)}$. $\boxed{\psi = \text{Im}(\Phi) = A(x^3 - 3xy^2)}$. $\boxed{(u, v) = \nabla\phi = A(3x^2 - 3y^2, -6xy).}$

(b) (Figure generated using R.) The wedge boundaries are straight lines given by the level curve $\psi = 0$. The stagnation point at the origin is shown. In the vector field, it was difficult to get a good picture with the arrows draw proportional to the vector's length, so we made them all the same size.



Vector field and streamlinesfor $\Phi(z) = Az^3$.

(c) This is similar to the previous problem. The rays at angles $\theta = 0, \pi/3, 2\pi/3, ..., 5\pi/3$ are all streamlines. So, they divide the plane into 6 wedges with vertex angle $\pi/3$. We can view each of the wedges as having walls at its boundary and the streamlines represent a flow within the wedge.

Problem 6. (10 points)
Suppose the vector field $\mathbf{F} = (u, v)$ is divergence free and irrotational (curl free). Show that u is a harmonic function

Solution: **Method 1.** We know that there is a complex potential function Φ with $\Phi' = u - iv$. Since Φ is analytic, so is Φ'. Since u is the real part of an analytic function, it is harmonic.

Method 2. Divergence and curl free mean

$$u_x + v_y = 0 \text{ and } v_x - u_y = 0.$$

Differentiating these equations by x and y respectively we have

$$u_{xx} + v_{yx} = 0 \text{ and } v_{xy} - u_{yy} = 0.$$

Now subtract these two equations to get $u_{xx} + u_{yy} = 0$ as needed.

Problem 7. (15: 5,5,5 points)
Let A be the unit disk. Assume it is made of a heat conducting material and that in our two dimensional world it only loses heat through its boundary. Then at steady state the temperature $T(x,y)$ in the disk is a harmonic function.

Suppose we hold the temperature of the boundary fixed at

$$T(e^{i\theta}) = T(\cos(\theta), \sin(\theta)) = \sin^2(\theta).$$

(a) *What is the temperature at the center of the disk?*

(b) *What is the maximum temperature on the disk?*

(c) *What is the minimum temperature on the disk?*

Challenge. *Find the temperature $T(x,y)$ throughout the disk.*

answers:

(a) We can use the mean value theorem for harmonic functions

$$T(0,0) = \frac{1}{2\pi}\int_0^{2\pi} T(e^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sin^2(\theta)\,d\theta = \boxed{\frac{1}{2}.}$$

(We'll leave the details of the last integration for you to supply.)

(b) The maximum value principle says T attains its maximum on the boundary. The maximum of $\sin^2(\theta)$ is $\boxed{1}$..

(c) Likewise, the minimum value principle says T attains its minimum on the boundary. The minimum of $\sin^2(\theta)$ is $\boxed{0}$..

Challenge. Method 1. The systematic way to solve this using the Poisson integral formula. You can look it up on the web or in the textbooks linked to from the class website. Computing the integral is doable, but a bit of a chore.

Method 2. Notice that

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2} = \mathrm{Re}\left(\frac{1-e^{i2\theta}}{2}\right)$$

Replacing $e^{i2\theta}$ by z^2 we have that, on the unit circle $z = e^{i\theta}$

$$T(z) = \mathrm{Re}((1-z^2)/2).$$

Since $(1-z^2)/2$ is analytic, its real part is harmonic over the whole disk, i.e.

$$T(x,y) = \mathrm{Re}\left(\frac{1-z^2}{2}\right) = \frac{1}{2} - \frac{x^2-y^2}{2}.$$

Problem 8. (10: 5,5 points) (Hints of Taylor and Laurent series.)
Consider the following infinite series.

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots$$

(a) *Remember your calculus and use the ratio test to say for what z the series converges.*

(b) *Let C be a large circle with center at the origin. What is the value of $\int_C f(z)\,dz$?*

answers:

(a) The limit of the ratio of (the absolute value of) successive terms in the series is

$$\lim_{n\to\infty} \frac{|1/z^{n+1}|}{|1/z^n|} = \lim_{n\to\infty} \frac{1}{|z|} = \frac{1}{|z|}.$$

So the series converges when

$$\frac{1}{|z|} < 1, \text{ i.e., when } \boxed{|z| > 1.}$$

(b) The only term in the series that has a non-zero integral is $1/z$. Therefore the integral is $2\pi i$.