18.04 Problem Set 6, Spring 2018

Calendar
Mar. 19-23: Reading: Topic 7 notes (Taylor and Laurent series)
Apr. 2-6: Reading: Topic 8 (Residue theorem)

Coming next

Apr. 9-13: Applications of the residue theorem.

Problem 1. (12 points)
Say whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1+2i}{1-i}\right)^n$ **(b)** $\displaystyle\sum_{n=0}^{\infty} i^n$ **(c)** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1-i}{1+2i}\right)^n$ **(d)** $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{10^n}$

Problem 2. (8 points)
Find the radius of convergence.

(a) $\displaystyle f_1(z)=\sum_{n=0}^{\infty}\frac{z^{3n}}{2^n}$ **(b)** $f_2(z)=1+3(z-1)+3(z-1)^2+(z-1)^3$

Problem 3. (8 points)

Suppose the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is R. Find the radius of convergence of each of the following.

(a) $\displaystyle\sum_{n=0}^{\infty} a_n z^{2n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} n^{-n} a_n z^n$

Problem 4. (10 points)
(a) Give a function f that is analytic in the punctured plane $(\mathbf{C}-\{1\})$, has a simple zero at $z=0$ and an essential singularity at $z=1$.

(b) Suppose f is analytic and has a zero of order m at z_0. Show that $g(z)=f'(z)/f(z)$ has a simple pole at z_0 with $\mathrm{Res}(g,z_0)=m$.

Problem 5. (20 points)

(a) What is the order of the pole of $\displaystyle f_1(z)=\frac{1}{(2\cos(z)-2+z^2)^2}$ at $z=0$.

Hint: Work with $1/f_1(z)$.

(b) Find the residue of $\displaystyle f_2(z)=\frac{z^2+1}{2z\cos(z)}$ at $z=0$.

(c) Let $\displaystyle f_3(z)=\frac{e^z}{z(z+1)^3}$. Find all the isolated singularities and compute the residue at each one.

(d) Find the residue at infinity of $\displaystyle f_4(z)=\frac{1}{1-z}$.

(e) Let $f_5(z) = \dfrac{\cos(z)}{\int_0^z f(w)\,dw}$, where $f(z)$ is analytic and $f(0) = 1$. Find the residue at $z = 0$.

Problem 6. (10 points)
Write the principal part of each function at the isolated singularity. Compute the corresponding residue.

(a) $f_1(z) = z^3 e^{1/z}$

(b) $f_2(z) = \dfrac{1 - \cosh(z)}{z^3}$

Problem 7. (8 points)
(a) Let $f(z) = (1+z)^a$, computed using the principal branch of log. Give the Taylor series around 0.

(b) Does the principal branch of $\sqrt{z}$ have a Laurent expansion in the domain $0 < |z|$?

Problem 8. (15 points)
Using variations of the geometric series find the following series expansions of

$$f(z) = \frac{1}{4 - z^2}$$

about $z_0 = 1$.

(a) The Taylor series. What is the radius of convergence?

(b) The Laurent series on $1 < |z - 1| < R_1$. What is R_1?

(c) The Laurent series for $|z - 1| > 3$.

Problem 9. (15 points)
(a) Use the residue theorem to compute $\displaystyle\int_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)}\,dz$.

(b) Evaluate $\displaystyle\int_{|z|=1} e^{1/z}\sin(1/z)\,dz$.

(c) Explain why Cauchy's integral formula can be viewed as a special case of the residue theorem.

Problem 10. (15 points)
In this problem we will compute $\displaystyle\sum_{-\infty}^{\infty} \frac{1}{n^2}$ using the residue theorem. The techniques learned here are general. In particular, the use of $\cot(\pi z)$ is fairly common.

(a) Let $\phi(z) = \pi\cot(\pi z) = \pi\dfrac{\cos(\pi z)}{\sin(\pi z)}$. At all the singular points give the order of the pole and the residue.

(b) Take the contour C_N which is the square with vertices at $\pm(N + 1/2) \pm i(N + 1/2)$. Use the Cauchy residue theorem to write an expression for

$$\int_{C_N} \frac{\pi\cot(\pi z)}{z^2}\,dz.$$

You'll need to do some work to compute the residue at $z = 0$.

(c) We'll tell you that $|\cot(\pi z)| < 2$ along the contour C_N. Use this to show that

$$\lim_{N\to\infty} \int_{C_N} \frac{\pi \cot(\pi z)}{z^2}\, dz = 0.$$

(d) Use parts (b) and (c) to compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Problems below here are not assigned. Do them just for fun.

Problem Fun 1. (No points)
By considering the 3 series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$, $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$, $\displaystyle\sum_{n=1}^{\infty} z^n$, show that a power series may converge on all, some or no points on the boundary of its disk of convergence.

Problem Fun 2. (No points)
Suppose that there exists a function $f(z)$ which is analytic at $z = 0$ and which satisfies the differential equation

$$(1 + z)f'(z) = 2f(z), \text{ with } f(0) = 1.$$

(a) Solve this equation to get a closed-form expression for $f(z)$.

(b) Find the formula for the power series coefficients of $f(z)$ directly from the differential equation.

(c) Check your answer to part(b) against the Taylor series obtained by expanding out the closed-form expression for the solution found in part (a).

Problem Fun 3. (No points) Show that $|\cot(\pi z)| < 2$ along the contour in problem 10.

Hint, show that along the vertical sides $|\cot(\pi z)| < 1$, while along the horizontal sides $|\cot(\pi z)| < 2$.

Problem Fun 4. (No points) Suppose the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is R. Show that the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} n^2 a_n z^n$ is also R.

18.04 Complex Variables with Applications
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