18.04 Problem Set 6, Spring 2018 Solutions

Problem 1. (12 points)

Say whether the following series converge or diverge.

(a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1+2i}{1-i}\right)^n$ **(b)** $\displaystyle\sum_{n=0}^{\infty} i^n$ **(c)** $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1-i}{1+2i}\right)^n$ **(d)** $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{10^n}$

answers: **(a)** This is a geometric series with ratio $r = \dfrac{1+2i}{1-i}$. Since $|r| = \dfrac{\sqrt{5}}{\sqrt{2}} > 1$, the series diverges.

(b) This is a geometric series with ratio $r = i$. Since $|r| = 1$, the series diverges. (We need the terms of the series to go to 0.)

(c) This is a geometric series with ratio $r = \dfrac{1-i}{1+2i}$. Since $|r| = \dfrac{\sqrt{2}}{\sqrt{5}} < 1$, the series converges.

(d) Using the ratio test we have

$$L = \lim_{n\to\infty}\frac{(n+1)!/10^{n+1}}{n!/10^n} = \lim_{n\to\infty}\frac{n+1}{10} = \infty$$

Since $L > 1$ the series diverges.

Problem 2. (8 points)

Find the radius of convergence.

(a) $f_1(z) = \displaystyle\sum_{n=0}^{\infty}\frac{z^{3n}}{2^n}$ **(b)** $f_2(z) = 1 + 3(z-1) + 3(z-1)^2 + (z-1)^3$

answers: **(a)** The series is a geometric series with ratio $\dfrac{z^3}{2}$. The series converges if $|z^3|/2 < 1$, i.e. for $\boxed{|z| < 2^{1/3}.}$

(b) This is a finite series. The radius of convergence is ∞.

Problem 3. (8 points)

Suppose the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ *is* R. *Find the radius of convergence of each of the following.*

(a) $\displaystyle\sum_{n=0}^{\infty} a_n z^{2n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} n^{-n} a_n z^n$

answers: **(a)** Let $w = z^2$. We know $\sum a_n w^n$ converges for $|w| < R$. That is, the series converges for $|z^2| < R$, equivalently for $|z| < R^{1/2}$. The radius of convergence is $R^{1/2}$.

(b) We'll see that the series converges for all z, i.e. the radius of convergence is infinite. The proof is by asymptotic comparison. Pick any z. For large enough n, know $|z|/n < R/2$. Thus, by asymptotic comparison to the convergent series $\sum |a_n|(R/2)^n$ the series converges for all z.

Problem 4. (10 points)

(a) *Give a function f that is analytic in the punctured plane ($\mathbf{C} - \{1\}$), has a simple zero at $z = 0$ and an essential singularity at $z = 1$.*

(b) *Suppose f is analytic and has a zero of order m at z_0. Show that $g(z) = f'(z)/f(z)$ has a simple pole at z_0 with $\text{Res}(g, z_0) = m$.*

answers: (a) $f(z) = z e^{1/(z-1)}$ will do.

(b) This is a matter of writing out the Taylor series

$$\begin{aligned} f(z) &= a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \ldots \\ &= a_m(z - z_0)^m \cdot g(z), \ \text{ where } g(z_0) = 1 \\ f'(z) &= m a_m (z - z_0)^{m-1} + (m+1)a_{m+1}(z - z_0)^m + \ldots \\ &= m a_m (z - z_0)^{m-1} \cdot h(z), \ \text{ where } h(z_0) = 1 \end{aligned}$$

So,

$$\frac{f'(z)}{f(z)} = \frac{m a_m (z - z_0)^{m-1} h(z)}{a_m (z - z_0)^m g(z)} = \frac{m}{z - z_0} \cdot \frac{h(z)}{g(z)}$$

Since $h(z)/g(z)$ is analytic and $h(z_0)/g(z_0) = 1$, the desired result $\text{Res}(g, z_0) = m$ follows.

Problem 5. (20 points)

(a) *What is the order of the pole of* $f_1(z) = \dfrac{1}{(2\cos(z) - 2 + z^2)^2}$ *at $z = 0$.*

Hint: Work with $1/f_1(z)$.

Solution: Let $g = 1/f_1 = (2\cos(z) - 2 + z^2)^2$. We write out the Taylor series for this

$$g(z) = \left(2\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right) - 2 + z^2\right)^2 = z^8\left(\frac{2}{4!} + a_9 z + + \ldots\right)$$

Since g has a zero of order 8, $f_1 = 1/g$ has a pole of order 8 at $z = 0$.

(b) *Find the residue of* $f_2(z) = \dfrac{z^2 + 1}{2z\cos(z)}$ *at $z = 0$.*

Solution: Since $\cos(0) = 1$, $g(z) = z f_2(z)$ is analytic at $z = 0$. This tells us the pole is simple and $\text{Res}(f_2, 0) = g(0) = 1/2$.

(c) *Let* $f_3(z) = \dfrac{e^z}{z(z+1)^3}$. *Find all the isolated singularities and compute the residue at each one.*

Solution: There are poles at $z = 0$ and $z = -1$.

At $z = 0$: the pole is simple,

$$\text{Res}(f_3, 0) = \lim_{z \to 0} z f_3(z) = 1 \ \text{ (by inspection).}$$

At $z=-1$: $g(z)=(z+1)^3 f_3(z)=\dfrac{e^z}{z}$ is analytic at $z=-1$. If $g(z)=a_0+a_1(z+1)+...$, then $\text{Res}(f,-1)=a_2=g''(-1)/2!$. Now it's easy to compute that $\text{Res}(f_3,-1)=-5e^{-1}/2$.

(d) *Find the residue at infinity of* $f_4(z)=\dfrac{1}{1-z}$.

Solution: First we find

$$g(z)=\frac{1}{w^2}f_4(1/w)=\frac{1}{w^2}\cdot\frac{1}{1-1/w}=\frac{1}{w(w-1)}$$

So

$$\boxed{\text{Res}(f_4,\infty)=-\text{Res}(g,0)=1.}$$

(e) *Let* $f_5(z)=\dfrac{\cos(z)}{\int_0^z f(w)\,dw}$, *where* $f(z)$ *is analytic and* $f(0)=1$. *Find the residue at* $z=0$.

Let $g(z)=\displaystyle\int_0^z f(w)\,dw$. So g is analytic and $g(0)=0$ and $g'(0)=f(0)=1$. That is g has a simple zero at $z=0$. Thus, $f_5(z)=\cos(z)/g(z)$ has a simple pole at $z=0$ and we have,

$$\text{Res}(f_5,0)=\frac{\cos(0)}{g'(0)}=1$$

Problem 6. (10 points)
Write the principal part of each function at the isolated singularity. Compute the corresponding residue.

(a) $f_1(z)=z^3e^{1/z}$

Solution: The only singularity is at $z=0$. We know

$$e^{1/z}=1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3!z^3}+\frac{1}{4!z^4}+...$$

So,

$$f_1(z)=z^3+z^2+\frac{z}{2}+\frac{1}{3!}+\frac{1}{4!z}+...$$

Thus, we have $\boxed{\text{Res}(f_1,0)=\dfrac{1}{24}.}$

(b) $f_2(z)=\dfrac{1-\cosh(z)}{z^3}$

Solution: The only singularity is at $z=0$. We know

$$\cosh(z)=1+\frac{z^2}{2}+\frac{z^4}{4!}+...$$

So,

$$f_2(z)=\frac{-z^2/2-z^4/4!-...}{z^3}=-\frac{1}{2z}-\frac{z}{4!}-...$$

Thus, we have $\boxed{\text{Res}(f_2,0)=-\dfrac{1}{2}.}$

Problem 7. (8 points)
(a) *Let $f(z) = (1+z)^a$, computed using the principal branch of* $\log$. *Give the Taylor series around 0.*

Solution: We'll do this using derivatives. Keeping the branch in mind $f(z) = \mathrm{e}^{a\log(1+z)}$. On the principle branch, $f(0) = 1$. Taking derivatives we have

$$f^{(n)}(z) = a(a-1)\cdots(a-n+1)(1+z)^{a-n}, \quad f^{(n)}(0) = a(a-1)\cdots(a-n+1).$$

So

$$f(z) = 1 + az + \frac{a(a-1)}{2}z^2 + \ldots + \frac{a(a-1)\cdots(a-n+1)}{n!}z^n + \ldots = \sum_{n=0}^{\infty}\binom{a}{n}z^n$$

where $\binom{a}{n}$ is defined as $\dfrac{a(a-1)\cdots(a-n+1)}{n!}$.

The question didn't ask for the following, but they are worth noting.

1. If $a = n$ is a nonnegative integer then the Taylor coefficients are 0 for powers bigger than n. For such a, f is entire and the radius of convergence is ∞.

2. For all other a the disk of convergence centered at 0, goes as far as the first singularity, which is at $z = 1$. That is, the radius of convergence is 1.

(b) *Does the principal branch of $\sqrt{z}$ have a Laurent expansion in the domain $0 < |z|$?*

Solution: No, $\sqrt{z}$ is not analytic on the region $0 < |z|$. In fact, it is not analytic on any annulus centered at 0.

Problem 8. (15 points)
Using variations of the geometric series find the following series expansions of

$$f(z) = \frac{1}{4-z^2}$$

about $z_0 = 1$.

(a) *The Taylor series. What is the radius of convergence?*

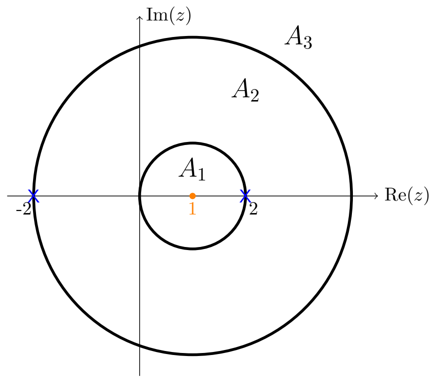
(b) *The Laurent series on $1 < |z-1| < R_1$. What is R_1?*

(c) *The Laurent series for $|z-1| > 3$.*

answers: Here is a picture showing the singularities of f and the various regions. The labels are:

$$A_1 : |z-1| < 1, \qquad A_2 : 1 < |z-1| < 3, \qquad A_3 : 3 < |z-1|.$$

We'll get a different Laurent series in each region.



The calculations will be easier if we express f using partial fractions

$$f(z) = \frac{1}{(2-z)(2+z)} = \frac{1}{4(2-z)} + \frac{1}{4(2+z)}.$$

We write the Laurent series for each piece in each region.

$\dfrac{1}{2-z} = \dfrac{1}{1-(z-1)}$.

In A_1 we have $|z-1| < 1$, so the geometric series

$$\frac{1}{2-z} = \frac{1}{1-(z-1)} = 1 + (z-1) + (z-1)^2 + ... = \sum_{n=0}^{\infty} (z-1)^n \tag{1}$$

converges.

In A_2 and A_3 we have $|z-1| > 1$, so the geometric series

$$\frac{1}{2-z} = -\frac{1}{z-1} \cdot \frac{1}{1-1/(z-1)} = -\sum_{n=1}^{\infty} \left(\frac{1}{z-1}\right)^n \tag{2}$$

converges.

Likewise $\dfrac{1}{2+z} = \dfrac{1}{3+(z-1)} = \dfrac{1}{3} \cdot \dfrac{1}{1+(z-1)/3}$.

In A_1 and A_2 we have $|z-1|/3 < 1$, so the geometric series

$$\frac{1}{2+z} = \frac{1}{3} \cdot \frac{1}{1+(z-1)/3} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{3}\right)^n \tag{3}$$

converges.

In A_3 we have $|z-1|/3 > 1$, so $3/|z-1| < 1$ and the geometric series

$$\frac{1}{2+z} = \left(\frac{1}{z-1}\right) \cdot \left(\frac{1}{1+3/(z-1)}\right) = \frac{1}{z-1} \sum_{n=0}^{\infty} \left(\frac{-3}{z-1}\right)^n = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{(z-1)^n} \tag{4}$$

converges.

We can now answer each part using $f(z) = \dfrac{1}{4}\left(\dfrac{1}{2-z} + \dfrac{1}{2+z}\right)$

(a) On A_1: $f(z)$ is analytic on A_1 and the Taylor series is

$$f(z) = \frac{1}{4}\left(\sum_{n=0}^{\infty}(z-1)^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{3^n}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{4} + \frac{(-1)^n}{12\cdot 3^n}\right)(z-1)^n$$

(b) On A_2: The Laurent series is

$$f(z) = -\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{(z-1)^n} + \frac{1}{12}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{3^n}$$

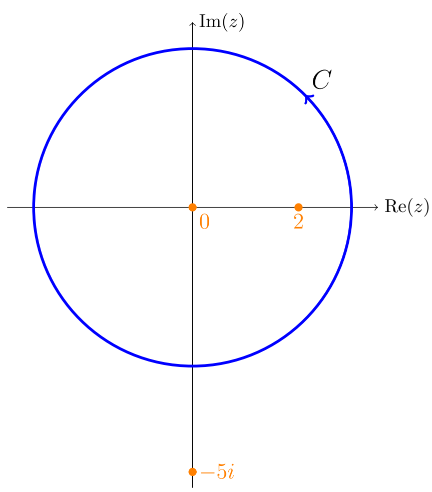
(c) On A_3: The Laurent series is

$$f(z) = -\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{(z-1)^n} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{(z-1)^n} = \frac{1}{4}\sum_{n=1}^{\infty}\left(-1+(-3)^{n-1}\right)\frac{1}{(z-1)^n}.$$

Problem 9. (15 points)

(a) *Use the residue theorem to compute* $\displaystyle\int_{|z|=3}\frac{e^{iz}}{z^2(z-2)(z+5i)}\,dz.$

Solution: The function $f(z) = \dfrac{e^{iz}}{z^2(z-2)(z+5i)}$ is meromorphic with poles at 0, 2, $-5i$. Of these, only 0 and 2 are inside the countour of integration C: $|z| = 3$.



So, $\displaystyle\int_C f(z)\,dz = 2\pi i(\text{Res}(f,0) + \text{Res}(f,2))$. To finish the problem we must compute the residues.

At $z=0$: $g(z) = z^2 f(z) = \dfrac{e^{iz}}{(z-2)(z+5i)}$ is analytic. Thus, $\text{Res}(f,0) = g'(0) = \dfrac{-12+5i}{100}$. (We'll leave it to you to provide the details for finding $g'(0)$.)

At $z=2$: $g(z) = (z-2)f(z) = \dfrac{e^{iz}}{z^2(z+5i)}$ is analytic. Thus, $\text{Res}(f,2) = g(2) = \dfrac{e^{2i}}{4(2+5i)}$.

We conclude that $\boxed{\displaystyle\int_{|z|=3}\frac{e^{iz}}{z^2(z-2)(z+5i)}\,dz = 2\pi i\left(\frac{-12+5i}{100} + \frac{e^{2i}}{4(2+5i)}\right)}$.

(b) *Evaluate* $\int_{|z|=1} e^{1/z} \sin(1/z)\, dz$.

Solution: The integrand $f(z) = e^{1/z}\sin(1/z)$ has a pole at $z = 0$ and no other singularities. To compute the residue we multiply series

$$f(z) = \left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots\right)\left(\frac{1}{z} - \frac{1}{3!z^3} + \ldots\right) = \frac{1}{z} + \frac{1}{z^2} + \ldots$$

Thus, $\mathrm{Res}(f, 0) = 1$ and $\int_{|z|=1} f(z)\, dz = \boxed{2\pi i.}$

(c) *Explain why Cauchy's integral formula can be viewed as a special case of the residue theorem.*

Solution: Cauchy's integral formula says: if C is a simple closed curve and f is analytic on and inside C and $\int_C \frac{f(z)}{z - z_0} f(z_0) = 2\pi i f(z_0)$.

On the other hand, since f is analytic the only singularity of $f(z)/(z - z_0)$ is at $z = z_0$. So, the residue theorem says $\int_C \frac{f(z)}{z - z_0} f(z_0) = 2\pi i\, \mathrm{Res}(f(z)/(z - z_0), z_0)$.

To see both theorems give the same result we note that z_0 is a simple pole and $\mathrm{Res}(f(z)/(z - z_0), z_0) = f(z_0)$.

Note: the condition f is analytic on C can be relaxed. It is enough the f be analytic inside C and continuous on and inside C. Even this can be further relaxed.

Problem 10. (15 points)

In this problem we will compute $\sum_{-\infty}^{\infty} \frac{1}{n^2}$ *using the residue theorem. The techniques learned here are general. In particular, the use of* $\cot(\pi z)$ *is fairly common.*

(a) *Let* $\phi(z) = \pi\cot(\pi z) = \pi\frac{\cos(\pi z)}{\sin(\pi z)}$. *At all the singular points give the order of the pole and the residue.*

Solution: We know that $g(z) = \sin(\pi z)$ has zeros at all integers n. Also, $g'(n) = \pi\cos(n\pi)$ Since this is not zero, the zeros are simple. Therefore the poles of ϕ are simple and

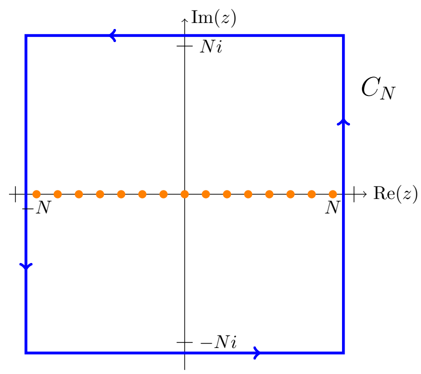
$$\mathrm{Res}(\phi, n) = \frac{\pi\cos(n\pi)}{\pi\cos(n\pi)} = 1.$$

(b) *Take the contour* C_N *which is the square with vertices at* $\pm(N + 1/2) \pm i(N + 1/2)$. *Use the Cauchy residue theorem to write an expression for*

$$\int_{C_N} \frac{\pi\cot(\pi z)}{z^2}\, dz.$$

You'll need to do some work to compute the residue at $z = 0$.

Solution:



C_N, with poles inside at the integers

First we compute the residues of f:
At $z = n \neq 0$: Since $1/n^2 \neq 0$, $\text{Res}(f, n) = \text{Res}(\phi, n)/n^2 = 1/n^2$.

At $z = 0$: Below we'll show that $\text{Res}(f, 0) = -\pi^2/3$.

The poles inside C_N are at $-N, -N+1, \ldots, 0, 1, 2, \ldots, N$. So, taking into account that the residue at $z = 0$ is special, we get

$$\int_{C_N} f(z)\,dz = 2\pi i \sum_{n=-N}^{N} \text{Res}(f, n) = 2\pi i \sum_{n=-N,\, n\neq 0}^{N} \frac{1}{n^2} - \frac{\pi^2}{3} = 2\sum_{n=1}^{N} \frac{1}{n^2} - \frac{\pi^2}{3}.$$

The last equality uses the fact $1/n^2 = 1/(-n)^2$.

The last thing we need to do is show how to compute the residue at $z = 0$. For the grunge work we'll work with $\cot(z)$. We can bring back the factors of π at the end. We know $\cot(z)$ has a simple pole at $z = 0$, so

$$\cot(z) = \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2$$

Because we'll be dividing by z^2 the residue will come from a_1. We compute this as follows:

$$\begin{aligned} \cot(z) &= \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 \\ &= \frac{\cos(z)}{\sin(z)} = \frac{1 - z^2/2 + \ldots}{z - z^3/3! + \ldots} \end{aligned}$$

Cross-multiplying we get

$$\begin{aligned} 1 - \frac{z^2}{2} + \ldots &= \left(\frac{b_1}{z} + a_0 + a_1 z + \ldots\right)\left(z - \frac{z^3}{3!} + \ldots\right) \\ &= b_1 + a_0 z + \left(a_1 - \frac{b_1}{3}\right) z^2 + \ldots \end{aligned}$$

Equating coefficients of z^n we get:
$1 = b_1$
$0 = a_0$
$-1/2 = a_1 - b_1/3!$, which implies $a_1 = -1/3$.

Thus $\cot(z) = \dfrac{1}{z} - \dfrac{z}{3} + \ldots$. This gives us

$$f(z) = \frac{\pi\cot(\pi z)}{z^2} = \frac{\pi}{z^2}\left(\frac{1}{\pi z} - \frac{\pi z}{3} + \ldots\right) = \frac{1}{z^3} - \frac{\pi^2}{3z} + \ldots$$

This shows $\text{Res}(f, 0) = -\pi^2/3$ as claimed above.

(c) *We'll tell you that $|\cot(\pi z)| < 2$ along the contour C_N. Use this to show that*

$$\lim_{N\to\infty} \int_{C_N} \frac{\pi\cot(\pi z)}{z^2}\,dz = 0.$$

Solution: The length of C_N is $2(2N+1)$. Since $|\cot(\pi z)| < 2$, along C_N we have

$$\left|\frac{\pi\cot(\pi z)}{z^2}\right| \le \frac{2\pi}{(N+1/2)^2}$$

So

$$\left|\int_{C_N} \frac{\pi\cot(\pi z)}{z^2}\,dz\right| \le \int_{C_N} \frac{2\pi}{(N+1/2)^2} d|z| = \frac{2\pi}{(N+1/2)^2}\cdot 4(2N+1).$$

This last expression clearly goes to 0 as N goes to infinity, so we have shown what we need to.

(d) *Use parts (b) and (c) to compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.*

Solution: By parts (b) and (c), letting $N \to \infty$, we have

$$\lim_{N\to\infty} \int_{C_n} f(z)\,dz = 2\pi i \left[2\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{\pi^2}{3}\right] = 0.$$

This implies $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.}$.

Problems below here are not assigned. Do them just for fun.

Problem Fun 1. (No points)

By considering the 3 series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$, $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$, $\displaystyle\sum_{n=1}^{\infty} z^n$, show that a power series may converge on all, some or no points on the boundary of its disk of convergence.

Solution: For all three series the radius of convergence is $R = 1$. So the boundary of the disk of convergence is the circle $|z| = 1$. (Often this is called the circle of convergence, which is a slightly confusing name as this problem shows.)

Since $\displaystyle\sum \frac{1}{n^2}$ is convergent, the series $\displaystyle\sum \frac{z^n}{n^2}$ converges absolutely everywhere on the circle $|z| = 1$.

Since $\displaystyle\sum \frac{(-1)^n}{n}$ converges (conditionally not absolutely) and $\displaystyle\sum \frac{1}{n}$ diverges. We see that the series $\displaystyle\sum \frac{z^n}{n}$ converges at some points on $|z| = 1$. (In fact, it turns out this series converges at every point on the circle except at $|z| = 1$.)

When $|z| = 1$ the terms in $\sum z^n$ do not decay to 0. Therefore the series is not convergent for any z on the unit circle.

Problem Fun 2. (No points)
Suppose that there exists a function $f(z)$ which is analytic at $z = 0$ and which satisfies the differential equation

$$(1+z)f'(z) = 2f(z), \quad \textit{with} \;\; f(0) = 1.$$

(a) *Solve this equation to get a closed-form expression for $f(z)$.*

Solution: The differential equation is separable: $f'/f = 2/(1+z)$. Solving we get $f(z) = C(z+1)^2$. The inital condition $f(0) = 1$ determines $C = 1$. So, $\boxed{f(z) = (z+1)^2.}$

(b) *Find the formula for the power series coefficients of $f(z)$ directly from the differential equation.*

Solution: We express $f(z)$ and $f'(z)$ as Taylor series

$$f(z) = a_0 + a_1 z + \ldots = \sum_{n=0}^{\infty} a_n z^n$$

$$f'(z) = a_1 + \ldots = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

Multiplying we get and substituting into the DE we get

$$(1+z)f'(z) = \sum_{n=0}^{\infty} (n a_n + (n+1)a_{n+1}) z^n = \sum 2a_n z^n.$$

Equation coefficients gives the relation: $2a_n = n a_n + (n+1)a_{n+1}$. A little algebra converts this to the recursive formula

$$a_{n+1} = \frac{(2-n)a_n}{1+n}.$$

The initial condition gives $f(0) = a_0 = 1$. Using the recursion relation we find $a_1 = 2$, $a_2 = 1$, $a_m = 0$ for $m \geq 3$. Thus, $f(z) = 1 + 2z + z^2$.

(c) *Check your answer to part(b) against the Taylor series obtained by expanding out the closed-form expression for the solution found in part (a).*

Solution: The answers to parts (a) and (b) are clearly the same.

Problem Fun 3. (No points) *Show that $|\cot(\pi z)| < 2$ along the contour in problem 10.*

Hint, show that along the vertical sides $|\cot(\pi z)| < 1$, while along the horizontal sides $|\cot(\pi z)| < 2$.

Solution: We know

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = i\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = \frac{i(e^{2\pi i z} + 1)}{e^{2i\pi i z} - 1}.$$

The right side of C_N is along the line $(N+1/2)+iy$. On this line

$$|\cot(\pi z)| = \left|\frac{\mathrm{e}^{i(2N+1)\pi}\mathrm{e}^{-2\pi y}+1}{\mathrm{e}^{i(2N+1)\pi}\mathrm{e}^{-2\pi y}-1}\right| = \left|\frac{-\mathrm{e}^{-2\pi y}+1}{-\mathrm{e}^{-2\pi y}-1}\right|.$$

Since $\mathrm{e}^{-2\pi y} > 0$, the denominator is clearly larger in magnitude than the numerator. So $|cot(\pi z)| < 1$ along the right side of C_N.

Since the left side of C_N is minus the right side and cot is an odd function, the result holds along the left side as well.

The top side of C_N is along the line $x + i(N+1/2)$. So along this line

$$|\cot(\pi z)| = \left|\frac{\mathrm{e}^{2\pi ix}\mathrm{e}^{-(2N+1)\pi}+1}{\mathrm{e}^{2\pi ix}\mathrm{e}^{-(2N+1)\pi}-1}\right| \le \frac{1+\mathrm{e}^{-(2N+1)\pi}}{1-\mathrm{e}^{-(2N+1)\pi}}$$

This is of the form $\dfrac{1+a}{1-a}$, with $0 < a \le \mathrm{e}^{-\pi}$. Since $(1+a)/(1-a)$ is an increasing function, the maximum is at $a = \mathrm{e}^{-\pi}$ and this is clearly less than 2 (in fact, less than 1.1).

Again, by symmetry the result holds on the bottom also.

We have shown that, along C_N, $|\cot(\pi z)| < 2$.

Problem Fun 4. (No points) *Suppose the radius of convergence of* $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ *is* R. *Show that the radius of convergence of* $\displaystyle\sum_{n=0}^{\infty} n^2 a_n z^n$ *is also* R.

Solution: Idea: if the we can use ratio test then the factor of n^2 does not change the limit of the ratio test. That is,

$$L = \lim_{m\to\infty} \frac{(n+1)^2|a_{n+1}z^{n+1}|}{n^2|a_n z^n|} = \lim \frac{(n+1)^2}{n^2} \lim \frac{|a_{n+1}z|}{|a_n|} = \lim \frac{|a_{n+1}z|}{|a_n|}.$$

Since we get the same limit with or without the factor of n^2, the radius of convergence is the same in both cases.

The problem is that the limit might not exist. We offer two more technical proofs.

Proof 1. Let $f(z) = \displaystyle\sum_{n=0} a_n z^n$. We know that $f(z)$ is anlaytic inside the radius of convergence R. By Taylor's theorem we also know that

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

has the same radius of convergence. Thus $g(z) = zf'(z) = \displaystyle\sum_{n=0}^{\infty} n a_n z^n$ also has radius of convergence R. Continuing in the same way, $zg'(z) = \displaystyle\sum_{n=0}^{\infty} n^2 a_n z^n$ has radius of convergence R.

Proof 2. Pick z with $|z| = r < R$. Then pick r_1 with $r < r_1 < R$. Since the original series has radius of convergence R, $\sum |a_n| r_1^n$ converges. Now, since $r_1/r > 1$, we know

$$\lim_{n\to\infty} \frac{n^2 r^n}{r_1^n} = \lim \frac{n^2}{(r_1/r)^n} = 0.$$

Thus $\sum |n^2 a_n z^n| = \sum n^2 |a_n| r^n$ converges by asymptotic comparison with $\sum |a_n| r_1^n$.
QED

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18.04 Complex Variables with Applications
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