18.04 Problem Set 7, Spring 2018

Calendar
Apr. 9-13: Topic 9 notes (definite integrals)

Coming next

Apr. 18-20: Conformal mapping

Problem 1. (21 points)

(a) Compute $\displaystyle\int_0^{2\pi} \frac{8\,d\theta}{5+2\cos(\theta)}$.

(b) Compute $\displaystyle\int_0^{2\pi} \frac{d\theta}{(3+2\cos(\theta))^2}$.

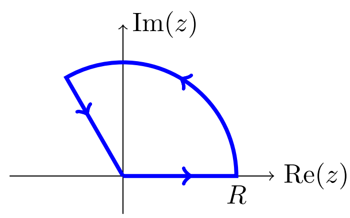
(c) Compute $\displaystyle\int_0^{2\pi} \frac{\sin^2(\theta)}{a+b\cos(\theta)}\,d\theta$, $a > |b| > 0$. (Answer: $\dfrac{2\pi}{b^2}(a-\sqrt{a^2-b^2})$.)

Problem 2. (21 points)

(a) Compute $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$. (Answer: π).

(b) Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$. (Answer: $\pi/3$.)

(c) Show $\displaystyle\int_0^{\infty} \frac{1}{x^3+1}\,dx = \frac{2\pi\sqrt{3}}{9}$ by integrating around the boundary of the circular sector shown and letting $R\to\infty$. The vertex angle of the sector is $2\pi/3$.



Circular sector with vertex angle $2\pi/3$.

Problem 3. (14 points)

(a) Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}\,dx$.

(b) Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{(x^2+1)^2}\,dx$. (Answer: $3\pi/(2e^2)$.)

Principal value.
Recall if $f(x)$ is continuous on the real axis except at, say, two points $x_1 < x_2$ then the

principal value of the integral along the entire x-axis is defined by

$$\text{p.v.} \int_{-\infty}^{\infty} = \lim \left[\int_{-R}^{x_1-r_1} f(x)\,dx + \int_{x_1+r_1}^{x_2-r_2} f(x)\,dx + \int_{x_2+r_2}^{R} f(x)\,dx. \right]$$

Here the limit is taken as $R \to \infty$, $r_1 \to 0$, $r_2 \to 0$. The extension to more points of discontinuity should be clear.

Problem 4. (14 points)

(a) Compute p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{e}^{3ix}}{x-2i}\,dx$.

(b) Derive the formula p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{x-w}\,dx = \begin{cases} \pi i \mathrm{e}^{iw} & \text{if } \operatorname{Im}(w) > 0 \\ -\pi i \mathrm{e}^{-iw} & \text{if } \operatorname{Im}(w) < 0. \end{cases}$

Problem 5. (14 points)

(a) Derive the formula p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{e}^{ix}}{(x-1)(x-2)}\,dx = \pi i(\mathrm{e}^{2i} - \mathrm{e}^{i})$.

(b) Derive the formula $\displaystyle\int_{0}^{\infty} \frac{\sin^2(x)}{x^2}\,dx = \pi/2$.

Hint: $\sin^2(x) = \dfrac{1-\cos(2x)}{2} = \dfrac{1}{2}\operatorname{Re}(1-\mathrm{e}^{2ix})$.

Problem 6. (7 points)

Compute $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{x^2+1}\,dx$. (Answer: $\pi/\sqrt{2}$.)

Problem 7. (15 points)

Let $f(x) = \begin{cases} 1 \text{ for } -1 < x < 1 \\ 0 \text{ elsewhere.} \end{cases}$

(a) (5) *Solution:* Compute the Fourier transform $\hat{f}(\omega) = \displaystyle\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\omega x}\,dx$.

(b) (10) Show that the formula for the Fourier inverse gives $f(x)$. That is, show

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{i\omega x}\,d\omega.$$

Hint: this will require an indented contour around 0.

Problems below here are not assigned. Do them just for fun.

Problem Fun.1. (No points)

(a) Let $f(x) = \mathrm{e}^{-x^2}$. Let $\omega > 0$ and $I = \displaystyle\int_{0}^{\infty} f(x)\mathrm{e}^{i2\omega x}\,dx$. Use the rectangle with vertices at 0, R, $R+i\omega$ and $i\omega$ and the known integral $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$ to show that $I = e^{-\omega^2}\sqrt{\pi}/2 + iB$. Here B is the imaginary part and we are not concerned with its value.

(b) Now use part (a) and symmetry to show that the Fourier transform $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx = \sqrt{\pi}e^{-\omega^2/4}$.

Problem Fun.2. (No points)
Compute $\int_0^{2\pi} (\cos\theta)^{2n}\,d\theta$. For $n = 1, 2, \ldots.$ $\left(\text{Answer: } \dfrac{2\pi\cdot(2n)!}{2^{2n}(n!)^2}.\right)$

Problem Fun.3. (No points)
Compute p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$.

Is this the same as the integral $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$ without the principal value?