18.04 Problem Set 7, Spring 2018 Solutions

Problem 1. (21 points)

(a) *Compute* $\displaystyle\int_0^{2\pi} \frac{8\,d\theta}{5+2\cos(\theta)}$.

Solution: Call the integral to compute I. We let $z = e^{i\theta}$ and convert the integral to a line integral over the unit circle. As usual, we have

$$d\theta = \frac{dz}{iz} \quad \text{and} \quad \cos(\theta) = \frac{z+1/z}{2} = \frac{z^2+1}{2z}$$

So

$$I = \int_{|z|=1} \frac{8}{5+2(z^2+1)/2z} \cdot \frac{dz}{iz} = \int_{|z|=1} \frac{8}{i(z^2+5z+1)}\,dz$$

Let $f(z) = \dfrac{8}{i(z^2+5z+1)}$. The poles of f are at $\dfrac{-5\pm\sqrt{21}}{2}$. Of these only $z_1 = (-5+\sqrt{21})/2$ is inside $|z| = 1$. So, by the residue theorem

$$I = 2\pi i \operatorname{Res}(f, z_1) = 2\pi i \lim_{z\to z_1} \frac{(z-z_1)8}{i(z^2+5z+1)} = \frac{16\pi}{2z_1+5} = \boxed{\frac{16\pi}{\sqrt{21}}.}$$

As often, the limit was computed using L'Hospital's rule. The fact that the limit exists implies the pole was simple and the limit is its residue.

(b) *Compute* $\displaystyle\int_0^{2\pi} \frac{d\theta}{(3+2\cos(\theta))^2}$.

Solution: Call the integral I. Using the computations from part (a) we get

$$I = \int_{|z|=1} \frac{1}{(3+2(z^2+1)/2z)^2} \cdot \frac{dz}{iz} = \int_{|z|=1} \frac{z}{i(z^2+3z+1)^2}\,dz.$$

Let $f(z) = \dfrac{z}{i(z^2+3z+1)^2}$. The poles of f are $z_1 = \dfrac{-3+\sqrt{5}}{2}$ and $z_2 = \dfrac{-3-\sqrt{5}}{2}$. Of these only z_1 is inside $|z| = 1$. So, by the residue theorem

$$I = 2\pi i \operatorname{Res}(f, z_1)$$

The pole is order 2, so we need to do a little work to compute it. Let

$$g(z) = (z-z_1)^2 f(z), \quad \text{so} \quad \operatorname{Res}(f, z_1) = g'(z_1).$$

Computing $g'(z_1)$ is not hard. It's probably easiest to factor the denominator symbolically using the roots z_1 and z_2.

$$g(z) = \frac{(z-z_1)^2 z}{i(z-z_1)^2(z-z_2)^2} = \frac{z}{i(z-z_2)^2}.$$

By direct computation using the quotient rule we have

$$g'(z_1) = \frac{(z_1-z_2)^2 - 2z_1(z_1-z_2)}{i(z_1-z_2)^4} = \frac{3\sqrt{5}}{25i}.$$

The calculation was done using $z_1 - z_2 = \sqrt{5}$. Thus $I = 2\pi i \,\text{Res}(f, z_1) = \boxed{\dfrac{6\pi\sqrt{5}}{25}}$.

(c) *Compute* $\displaystyle\int_0^{2\pi} \frac{\sin^2(\theta)}{a + b\cos(\theta)}\, d\theta, \quad a > |b| > 0.$ *(Answer:* $\dfrac{2\pi}{b^2}(a - \sqrt{a^2 - b^2})$*.)*

Solution: As usual, call the integral in question I. The conversion of this integral to one over the unit circle is similar to the previous parts. On the unit circle $z = e^{i\theta}$, so:

$$\sin^2(\theta) = \left(\frac{z - 1/z}{2i}\right)^2 = \frac{(z^2-1)^2}{-4z^2}.$$

After a little algebra we have $\displaystyle I = \int_{|z|=1} f(z)\, dz$, where

$$f(z) = \frac{\sin^2(\theta)}{a + b\cos(\theta)} \cdot \frac{1}{iz} = \frac{(z^2-1)^2}{-2iz^2(bz^2 + 2az + b)} = \frac{(z^2-1)^2}{-2ibz^2(z^2 + 2az/b + 1)}.$$

The poles of $f(z)$ are at 0, $z_1 = \dfrac{-a + \sqrt{a^2-b^2}}{b}$ and $z_2 = \dfrac{-a - \sqrt{a^2-b^2}}{b}$. Only 0 and z_1 are inside the curve $|z| = 1$. So,

$$I = 2\pi i(\text{Res}(f, 0) + \text{Res}(f, z_1)).$$

All that's left is to slog through computing the residues.

At $z = 0$: Let $g(z) = z^2 f(z) = \dfrac{(z^2-1)^2}{-2ib(z^2 + 2az/b + 1)}$. So, $\text{Res}(f, 0) = g'(0) = a/ib^2$. (This last value is not hard to compute.)

At $z = z_1$: Since the pole is simple this is not too hard to do by brute force. Here's a somewhat more delicate way of doing the computation. Factoring the denominator using the poles z_1 and z_2, we have

$$f(z) = \frac{(z^2-1)^2}{-2ibz^2(z - z_1)(z - z_2)}.$$

So, $\text{Res}(f, z_1) = \dfrac{(z_1^2-1)^2}{-2ibz_1^2(z_1 - z_2)}$. It is easy to see that $z_1 z_2 = 1$ and $z_1 - z_2 = 2\sqrt{a^2-b^2}/b$. So, multiplying top and bottom by z_2^2 we get

$$\text{Res}(f, z_1) = \frac{z_2^2}{z_2^2} \cdot \frac{(z_1^2-1)^2}{-2ibz_1^2(z_1 - z_2)} = \frac{(z_2 z_1^2 - z_2)^2}{-2ibz_2^2 z_1^2(z_1 - z_2)} = \frac{(z_1 - z_2)^2}{-2ib(z_1 - z_2)} = \frac{(z_1 - z_2)}{-2ib} = \frac{\sqrt{a^2-b^2}}{ib^2}.$$

Thus,

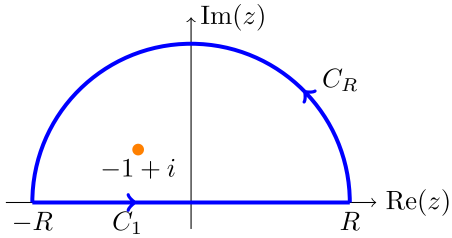
$$I = 2\pi i\left(\frac{a}{ib^2} - \frac{\sqrt{a^2-b^2}}{ib^2}\right) = \frac{2\pi(a - \sqrt{a^2-b^2})}{b^2}$$

as claimed.

Problem 2. (21 points)

(a) *Compute* $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$. *(Answer: π).*

Solution: Let I be the integral in question. Let $f(z) = \dfrac{1}{z^2+2z+2}$. Since the denominator decays like $1/z^2$ we can use a semicircular contour.



The residue theorem implies $\displaystyle\int_{C_1+C_R} f(z)\,dz = 2\pi i \sum$ residues of f inside the contour. We examine each of the pieces in this equation.

$\displaystyle\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$ by Theorem 9.1 in the Topic 9 notes.

$\lim_{R\to\infty} \int_{C_1} f(z)\,dx = I$ (this is clear).

So letting $R \to \infty$ we have $I = 2\pi i \sum$ residues of f in the upper half-plane.

The poles of f are at $z_1 = -1+i$ and $z_2 = -1-i$. Only z_1 is in the upper half-plane. The pole is simple so,

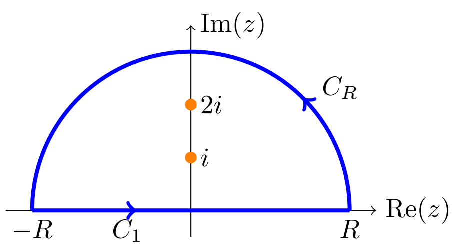
$$\mathrm{Res}(f, z_1) = \lim_{z\to z_1} (z-z_1) f(z) = \frac{1}{2z_1+2} = \frac{1}{2i}.$$

Thus $I = 2\pi i \cdot \dfrac{1}{2i} = \boxed{\pi}$.

(b) *Compute* $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx.$ *(Answer: $\pi/3$.)*

Solution: Let I be the integral in question. And let $f(z) = \dfrac{z^2}{(z^2+1)(z^2+4)}$. We proceed exactly as in part (a). Using the contour shown below we find

$$I = 2\pi i \sum \text{ residues of } f \text{ in the upper half-plane.}$$



Contour for part (b).

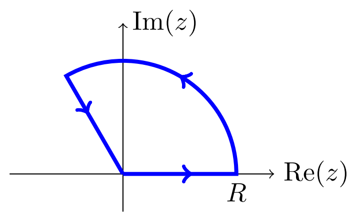
The poles of f are at $\pm i$ and $\pm 2i$. Only i and $2i$ are in the upper half-plane. All we need to do now is compute their residues.

$$\begin{aligned}\mathrm{Res}(f,i) &= \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{(z-i)z^2}{(z^2+1)(z^2+4)} = \frac{i^2}{2i(i^2+4)} = \frac{-1}{6i}\\ \mathrm{Res}(f,2i) &= \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}\frac{(z-2i)z^2}{(z^2+1)(z^2+4)} = \frac{4i^2}{4i(4i^2+1)} = \frac{1}{3i}.\end{aligned}$$

In both cases we computed the limit using L'Hospital's rule. Since the limit existed we know the pole is simple and the limit is the residue. This gives

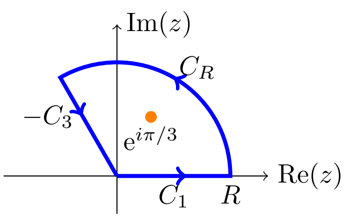
$$I = 2\pi i\left(-\frac{1}{6i}+\frac{1}{3i}\right) = \frac{\pi}{3}.$$

(c) *Show* $\displaystyle\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{2\pi\sqrt{3}}{9}$ *by integrating around the boundary of the circular sector shown and letting* $R\to\infty$*. The vertex angle of the sector is* $2\pi/3$*.*



Circular sector with vertex angle $2\pi/3$*.*

Solution: Let I be the integral in question. And let $f(z) = \dfrac{1}{z^3+1}$. We proceed similarly to parts (a) and (b). Here the contour is the circular sector shown. As usual, we put signs on the pieces of the contour that make the parametrization easier.



The poles of f are at $e^{i\pi/3}$, -1, $e^{-i\pi/3}$. The only one inside the contour is $z_1 = e^{i\pi/3}$. So

$$\int_{C_1+C_R-C_3} f(z)\,dz = 2\pi i\,\mathrm{Res}(f,z_1).$$

Looking at each segment of the curve in turn we have

On C_1: $\displaystyle\lim_{\mathbf{R}\to\infty}\int_{C_1} f(z)\,dz = \int_0^\infty \frac{1}{x^3+1}\,dx = I.$

On C_3: Parametrize the curve by $\gamma(t) = te^{i2\pi/3}$, where t runs from 0 to R. So

$$\lim_{\mathbf{R}\to\infty}\int_{C_3} f(z)\,dz = \int_0^\infty \frac{1}{t^3+1}e^{i2\pi/3}\,dt = e^{i2\pi/3}I.$$

Since $|f(z)| \approx 1/|z|^3$, the same argument as for semicircles shows that

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0.$$

Putting it together we have

$$\lim_{R\to\infty} \int_{C_1+C_R-C_3} f(z)\, dz = (1 - \mathrm{e}^{i2\pi/3})I = 2\pi i \,\mathrm{Res}(f, z_1).$$

All that's left is to compute the residue and do some algebra.

$$\mathrm{Res}(f, z_1) = \lim_{z\to z_1} \frac{z - z_1}{z^3 + 1} = \frac{1}{3z_1^2} = \frac{1}{3\mathrm{e}^{i2\pi/3}}.$$

So,

$$\begin{aligned} I &= \frac{2\pi i}{3\mathrm{e}^{2\pi i/3}(1 - \mathrm{e}^{i2\pi/3})} = \frac{2\pi i}{3(\mathrm{e}^{2\pi i/3} - \mathrm{e}^{i4\pi/3})} \\ &= \frac{2\pi i}{3(\mathrm{e}^{i2\pi/3} - \mathrm{e}^{-i2\pi/3})} \qquad\qquad (\text{use } \mathrm{e}^{i4\pi/3} = \mathrm{e}^{-i2\pi/3}) \\ &= \frac{\pi}{3(\mathrm{e}^{i2\pi/3} - \mathrm{e}^{-i2\pi/3})/2i} = \frac{\pi}{3\sin(2\pi/3)} = \frac{2\pi}{3\sqrt{3}}. \end{aligned}$$

Problem 3. (14 points)

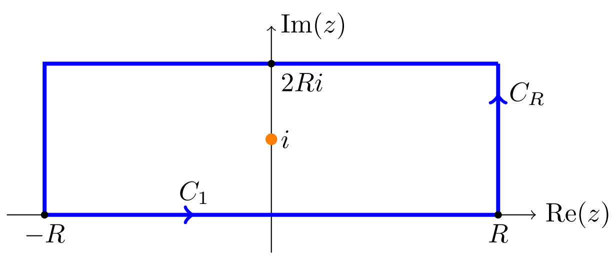
(a) *Compute* $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}\, dx.$

Solution: Let I be the integral in question. We start with complex replacement: Let

$$\tilde{I} = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i2x}}{x^2+1}\, dx.$$

So, $I = \mathrm{Re}(\tilde{I})$.

Consider the following contour



Now, let $f(z) = \frac{\mathrm{e}^{i2z}}{z^2+1}$. Theorem 9.2 in the notes (we could also use Theorem 9.1 here) implies that

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0.$$

The only pole of f inside the contour is $z = i$. So, as usual, we find

$$\tilde{I} = 2\pi i \operatorname{Res}(f, i) = 2\pi i \cdot \frac{\mathrm{e}^{-2}}{2i} = \mathrm{e}^{-2}\pi.$$

Finally: $\boxed{I = \operatorname{Re}(\tilde{I}) = \mathrm{e}^{-2}\pi.}$.

(b) *Compute* $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{(x^2+1)^2}\,dx$. *(Answer:* $3\pi/(2e^2)$.*)*

Solution: This is nearly identical to part (a). In this problem the denominator is squared. So very briefly, $f(z) = \dfrac{\mathrm{e}^{i2z}}{(z^2+1)^2}$ and $I = \operatorname{Re}(\tilde{I})$, where $\tilde{I} = 2\pi i \operatorname{Res}(f, i)$.

To compute the residue we make the following calculation:

$$g(z) = (z-i)^2 f(z) = \frac{(z-i)^2 \mathrm{e}^{i2z}}{(z^2+1)} = \frac{\mathrm{e}^{i2z}}{(z+i)^2}, \qquad \text{and} \qquad \operatorname{Res}(f, i) = g'(i).$$

This calculation is straightforward, we get $g'(i) = -\dfrac{3i\mathrm{e}^{-2}}{4}$, so $\tilde{I} = \dfrac{3\pi\mathrm{e}^{-2}}{2}$. Finally, $I = \operatorname{Re}(\tilde{I}) = \boxed{\dfrac{3\pi\mathrm{e}^{-2}}{2}.}$

Principal value.

Recall if $f(x)$ *is continuous on the real axis except at, say, two points* $x_1 < x_2$ *then the principal value of the integral along the entire* x*-axis is defined by*

$$p.v.\int_{-\infty}^{\infty} = \lim\left[\int_{-R}^{x_1-r_1} f(x)\,dx + \int_{x_1+r_1}^{x_2-r_2} f(x)\,dx + \int_{x_2+r_2}^{R} f(x)\,dx.\right]$$

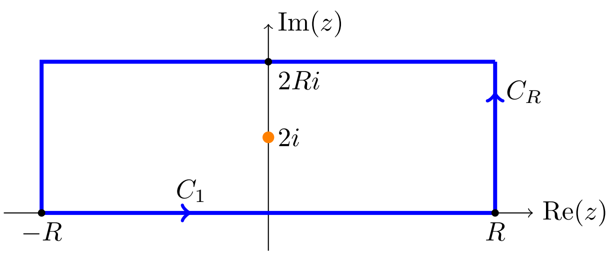
Here the limit is taken as $R \to \infty$, $r_1 \to 0$, $r_2 \to 0$. *The extension to more points of discontinuity should be clear.*

Problem 4. (14 points)

(a) *Compute p.v.* $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{e}^{3ix}}{x-2i}\,dx$.

Solution: In this case, the integrand has no poles along the real axis. So, the principal value only requires that we integrate over a symmetric interval $[-R, R]$ and let R go to infinity.

Let $f(z) = \dfrac{\mathrm{e}^{3iz}}{z-2i}$. f has one pole at $z = 2i$. The residue is easy to compute: $\operatorname{Res}(f, 2i) = \mathrm{e}^{-6}$.



The residue theorem implies

$$\int_{C_1+C_R} f(z)\,dz = 2\pi i\,\mathrm{Res}(f,2i) = 2\pi i \mathrm{e}^{-6}. \tag{1}$$

Looking at each piece:

$\displaystyle\lim_{R\to 0}\int_{C_R} f(z)\,dz = 0$ (Theorem 9.2 in the Topic 9 notes).

$\displaystyle\lim_{R\to \infty}\int_{C_1} f(z)\,dz = \text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{3ix}}{x-2i}\,dx$. (This is obvious).

Thus, letting $R\to\infty$ in Equation 1, we have

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{3ix}}{x-2i}\,dx = 2\pi i\mathrm{e}^{-6}.$$

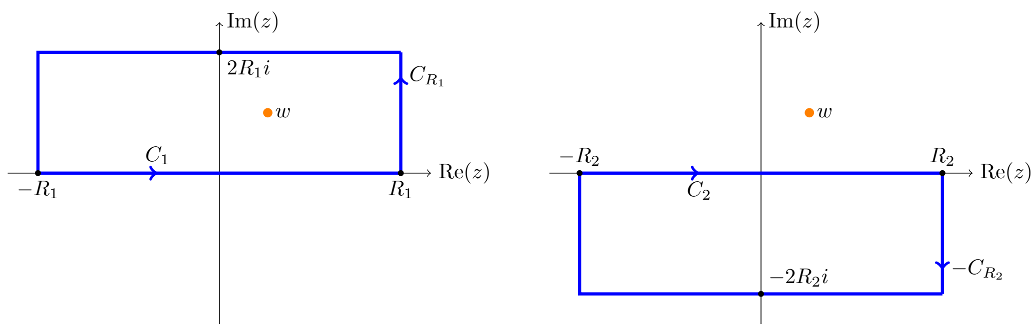
(b) *Derive the formula* $\displaystyle \textit{p.v.}\int_{-\infty}^{\infty}\frac{\cos(x)}{x-w}\,dx = \begin{cases}\pi i e^{iw} & \textit{if } \mathrm{Im}(w)>0\\ -\pi i\mathrm{e}^{-iw} & \textit{if } \mathrm{Im}(w)<0.\end{cases}$

Solution: Because w is complex our complexification trick is not going to work. Instead we work with the formula $\cos(x) = \dfrac{\mathrm{e}^{ix}+\mathrm{e}^{-ix}}{2}$. Using this formula we can break the integral into two pieces.

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{\cos(x)}{x-w}\,dx = \text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ix}+\mathrm{e}^{-ix}}{2(x-w)}\,dx = \text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ix}}{2(x-w)}\,dx + \text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-ix}}{2(x-w)}\,dx \tag{2}$$

We'll apply the residue theorem on a different contour for each piece.

Let $f_1(z) = \dfrac{\mathrm{e}^{iz}}{2(z-w)}$ and $f_2(z) = \dfrac{\mathrm{e}^{-iz}}{2(z-w)}$. Consider the following two contours.



We can work with the contour on the left with f_1 (Topic 9 notes, Theorem 9.2a) and with the contour on the right for f_2 (same theorem part b). Paying attention to the sign in the exponents. These theorems imply

$$\lim_{R_1\to\infty}\int_{C_{R_1}} f_1(z)\,dz = 0, \qquad \lim_{R_2\to\infty}\int_{C_{R_2}} f_2(z)\,dz = 0.$$

The case $\mathrm{Im}(w)>0$**.**

For now make the assumption $\text{Im}(w) > 0$. Both f_1 and f_2 have their only pole at $z = w$. Since $\text{Im}(w) > 0$, w is inside the left-hand contour and the right-hand contour contains no poles. So,

$$\int_{C_1 + C_{R_1}} f_1(z)\,dz = 2\pi i\,\text{Res}(f_1, w) = \pi i e^{iw}$$
$$\int_{C_1 - C_{R_2}} f_2(z)\,dz = 0.$$

Now we finish the problem in the usual manner: letting R_1 and R_2 go to infinity we have

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx = \text{p.v.} \int_{-\infty}^{\infty} f_1(x)\,dx + \text{p.v.} \int_{-\infty}^{\infty} f_2(x)\,dx = \pi i e^{iw} + 0 = \pi i e^{iw}.$$

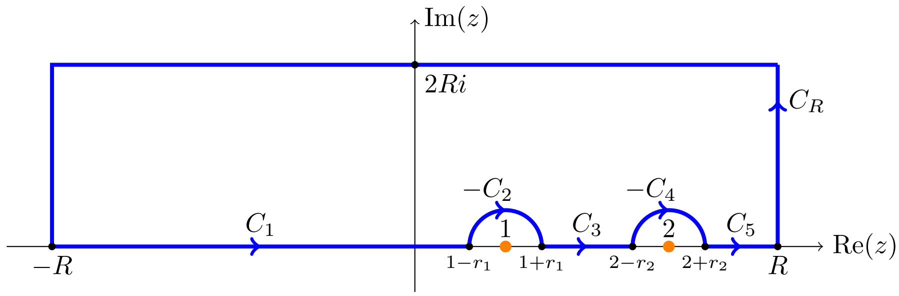
This is exactly what we were supposed to show.

The case $\text{Im}(w) < 0$ is the same. The minus sign occurs because the curve $C_1 + C_{R_2}$ is traversed in a clockwise direction.

Problem 5. (14 points)

(a) *Derive the formula p.v.* $\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-1)(x-2)}\,dx = \pi i(e^{2i} - e^{i})$.

Solution: Call the principal value in question I. Let $f(z) = \dfrac{e^{iz}}{(z-1)(z-2)}$. Consider the following contour



First we'll look at the integrals over each piece.

On C_R: $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$, by Theorem 9.2 in the Topic 9 notes.

On C_2: $\lim_{r_1\to 0} \int_{C_2} f(z)\,dz = \pi i\,\text{Res}(f, 1) = -\pi i e^{i}$. f has a simple pole at $z = 1$. So, this followis using Topic 9 Theorem 9.13 on integrating over a semicircle around a simple pole.

On C_4: $\lim_{r_2\to 0} \int_{C_4} f(z)\,dz = \pi i\,\text{Res}(f, 2) = \pi i e^{2i}$. The reasoning is the same as for C_2.

On $C_1 + C_3 + C_5$: Clearly $\lim \int_{C_1+C_3+C_5} f(z)\,dz = \text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx$. Here the limit is taken as $R \to \infty$ and r_1, $r_2 \to 0$.

Since there are no poles of $f(z)$ inside the closed contour we have

$$\int_{C_1 - C_2 + C_3 - C_4 + C_5 + C_R} f(z)\, dz = 0.$$

So, taking limits and doing a little algebra this becomes

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\, dx = \pi i(\mathrm{e}^{2i} - \mathrm{e}^{i}). \qquad QED$$

(b) *Derive the formula* $\displaystyle\int_0^\infty \frac{\sin^2(x)}{x^2}\, dx = \pi/2.$

Hint: $\sin^2(x) = \dfrac{1-\cos(2x)}{2} = \dfrac{1}{2}\,\mathrm{Re}(1 - \mathrm{e}^{2ix}).$

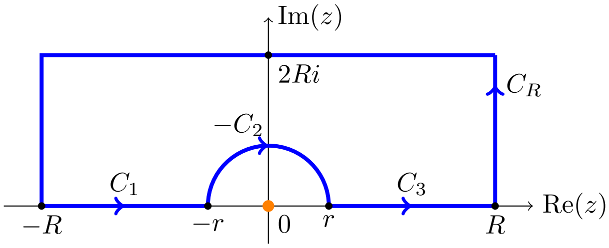
Solution: Call the integral in question I. First note that there is not problem at $x = 0$ since the integrand is continuous there. There is also no problem at ∞ because $\displaystyle\int_1^\infty 1/x^2\, dx$ converges. We start by using symmetry to get

$$2I = \text{p.v.} \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\, dx.$$

Since the integral is known to converge the principal value will give the same value and is convenient to use with indented contours.

Now, follow the hint and let $f(z) = \dfrac{1 - \mathrm{e}^{i2z}}{2z^2}$. Note that since the numerator is 0 at $z = 0$, f has a simple pole at $z = 0$.

We use the indented contour shown.



We'll make our argument a little more quickly than in previous problems. By Cauchy's theorem

$$\int_{C_1 - C_2 + C_3 + C_R} f(z)\, dx = 0.$$

By the usual limit theorems

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0 \quad \text{and} \quad \lim_{r \to 0} \int_{C_2} f(z)\, dz = \pi i\, \mathrm{Res}(f, 0)$$

By definition

$$\lim_{R\to\infty,\, r\to 0} \int_{C_1 + C_3} f(z)\, dz = \text{p.v.} \int_{-\infty}^{\infty} f(x)\, dx.$$

Computing $\mathrm{Res}(f,0)$:

$$f(z) = \frac{1-(1+i2z-4z^2/2-\ldots)}{2z^2} = -\frac{i}{z}+z+\ldots$$

So, $\mathrm{Res}(f,0) = -i$.

Putting it all togother: $2I = \text{p.v.} \int_{-\infty}^{\infty} f(x)\,dx = \pi i \,\mathrm{Res}(f,0) = \pi$. So, $\boxed{I = \pi/2}$,

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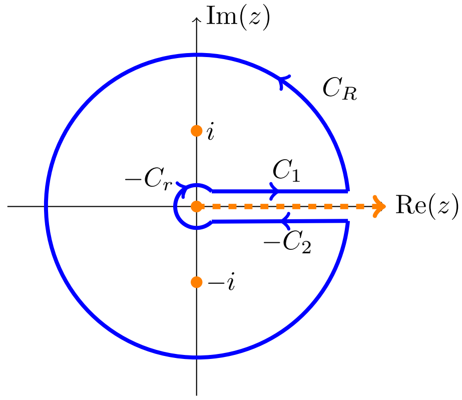
Problem 6. (7 points)

Compute $\int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx$. *(Answer:* $\pi/\sqrt{2}$.*)*

Solution: As always, call the integral I. We use the contour



Here the inner circle has radius r and the outer circle has radius R.

Let $f(z) = \frac{\sqrt{z}}{z^2+1}$. For f we make a branch cut along the positive real axis and use the branch with $0 < \arg(z) < 2\pi$.

Inside the contour f has poles at $\pm i$. So,

$$\int_{C_1+C_R-C_2-C_r} f(z)\,dz = 2\pi i(\mathrm{Res}(f,i)+\mathrm{Res}(f,-i)). \qquad (3)$$

We look at the pieces of the contour separately.

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0 \qquad \text{(Usual reason using Theorem 9.1.)}$$

$$\lim_{r\to 0}\int_{C_3} f(z)\,dz = 0 \qquad \text{(On } C_3 \text{ we have } |f(z)| = \frac{r^{1/2}}{|1+z^2|} \text{ is small for small } r.)$$

$$\lim_{R\to\infty,\, r\to 0}\int_{C_1} f(z)\,dz = \int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx = I \qquad \text{(Since, on } C_1,\ \arg(z)\approx 0, \text{ so } \sqrt{z}\approx\sqrt{x} \text{ and } f(z)\approx f(x).)$$

$$\lim_{R\to\infty,\, r\to 0}\int_{C_2} f(z)\,dz = -\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx = -I \qquad \text{(Since, on } C_2,\ \arg(z)\approx 2\pi, \text{ so } \sqrt{z}\approx-\sqrt{x} \text{ and } f(z)\approx -f(x).)$$

Taking the limit in Equation 3, this gives

$$2I = 2\pi i(\text{Res}(f, i) + \text{Res}(f, -i)).$$

All that's left is to compute the residues. Since $f(z) = \dfrac{\sqrt{z}}{(z+i)(z-i)}$, the residues are

$$\text{Res}(f, i) = \frac{\sqrt{i}}{2i} = \frac{\left(\mathrm{e}^{i\pi/2}\right)^{1/2}}{2i} = \frac{\mathrm{e}^{i\pi/4}}{2i} = \frac{1+i}{2\sqrt{2}i}$$

$$\text{Res}(f, -i) = \frac{\sqrt{-i}}{-2i} = \frac{\left(\mathrm{e}^{i3\pi/2}\right)^{1/2}}{-2i} = \frac{\mathrm{e}^{i3\pi/4}}{2i} = \frac{1-i}{2\sqrt{2}i}$$

So $2I = 2\pi i\left(\dfrac{1+i}{2\sqrt{2}i} + \dfrac{1-i}{2\sqrt{2}i}\right) = \dfrac{2\pi}{\sqrt{2}}$. Finally, we have $\boxed{\displaystyle\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx = \frac{\pi}{\sqrt{2}}.}$.

Problem 7. (15 points)
Let $f(x) = \begin{cases} 1 \text{ for } -1 < x < 1 \\ 0 \text{ elsewhere.} \end{cases}$

(a) (5) *Solution: Compute the Fourier transform* $\hat{f}(\omega) = \displaystyle\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\omega x}\,dx$.

Solution: We just compute the integral

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\omega x}\,dx = \int_{-1}^{1} \mathrm{e}^{-i\omega x}\,dx \\ &= \left.\frac{\mathrm{e}^{-i\omega x}}{-i\omega}\right|_{-1}^{1} = \boxed{\frac{\mathrm{e}^{i\omega} - \mathrm{e}^{-i\omega}}{iw}} = \frac{2\sin(\omega)}{\omega}.\end{aligned}$$

(b) (10) *Show that the formula for the Fourier inverse gives* $f(x)$*. That is, show*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{i\omega x}\,d\omega.$$

Hint: this will require an indented contour around 0*.*

Solution: We want to show: $f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{f}(\omega)\,d\omega$. We start computing:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega} - \mathrm{e}^{-i\omega}}{i\omega}\,\mathrm{e}^{i\omega x}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega(x+1)} - \mathrm{e}^{i\omega(x-1)}}{i\omega}\,d\omega.$$

Since the entire integral converges, we get the same result if we compute its principal value. The advantage is that we can compute the principal value of each piece separately!

Here are the results for each piece. We derive the results below.

$$\frac{1}{2\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega(x+1)}}{iw}\,d\omega = \begin{cases} -1/2 & \text{for } x < -1 \\ 1/2 & \text{for } x > -1 \end{cases} \tag{4}$$

$$\frac{1}{2\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\omega(x-1)}}{iw}\,d\omega = \begin{cases} -1/2 & \text{for } x < 1 \\ 1/2 & \text{for } x > 1 \end{cases} \tag{5}$$

Subtracting these two pieces we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i\omega(x+1)}}{iw}\,d\omega-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i\omega(x-1)}}{iw}\,d\omega=\begin{cases}0 & \text{for } x<-1\\ 1 & \text{for } -1<x<1\\ 0 & \text{for } 1<x\end{cases}$$

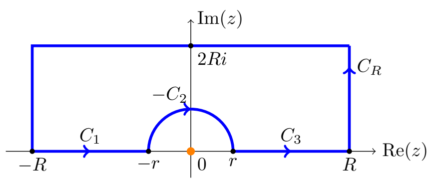
This is exactly $f(x)$ as claimed.

All that's left is to use contour integration to prove Equations 4 and 5. We will do this quickly. It is nearly identical to Example 9.15 in the Topic 9 notes.

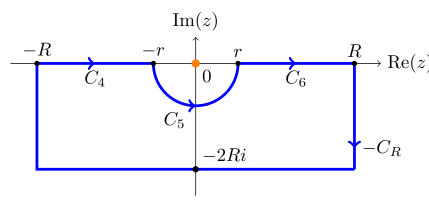
Let $f_a(z)=\dfrac{\mathrm{e}^{iaz}}{iz}$.

We start by assuming that $a>0$ and use the indented contour shown below on the left. The integrand has no poles inside the contour so

$$\int_{C_1-C_2+C_3+C_R} f_a(z)\,dz=0.$$



Contour for $a>0$



Contour for $a<0$

Next we break the contour into pieces.

$$\lim_{R\to\infty,\,r\to 0}\int_{C_1+C_3} f_a(z)\,dz=\text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ia\omega}}{i\omega}\,d\omega. \qquad \text{(This is clear.)}$$

$$\lim_{R\to\infty}\int_{C_R} f_a(z)\,dz=0. \qquad \text{(Theorem 9.2 in the Topic 9 notes.)}$$

$$\lim_{r\to 0}\int_{C_2} f_a(z)\,dz=\pi i\,\mathrm{Res}\left(\frac{\mathrm{e}^{iaz}}{iz},0\right)=\pi \qquad \text{(Theorem 9.13)}$$

Combining all this together we have, for $a>0$

$$\lim\int_{C_1-C_2+C_3+C_R}\frac{\mathrm{e}^{iaz}}{iz}\,dz=\text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{aiz}}{iz}-\pi=0 \qquad \text{i.e. for } a>0\quad \frac{1}{2\pi}\text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ai\omega}}{i\omega}\,d\omega=\frac{1}{2}. \tag{6}$$

Now assume $a<0$. Using the contour above on the right, we find in exactly the same way that

$$\lim\int_{C_4+C_5+C_6-C_R}\frac{\mathrm{e}^{iaz}}{iz}\,dz=\text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{aiz}}{iz}+\pi=0 \qquad \text{i.e. for } a<0\quad \frac{1}{2\pi}\text{p.v.}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ai\omega}}{i\omega}\,d\omega=-\frac{1}{2}. \tag{7}$$

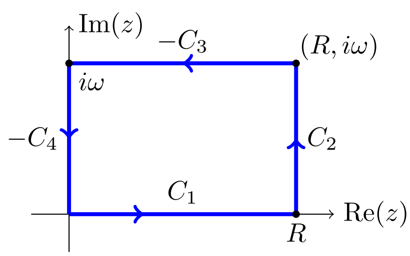
Letting $a=x+1$ or $a=x-1$. Equations 6 and 7 prove Equations 4 and 5. We're done!

Problems below here are not assigned. Do them just for fun.

Problem Fun.1. (No points)

(a) *Let* $f(x) = \mathrm{e}^{-x^2}$. *Let* $\omega > 0$ *and* $I = \int_0^\infty f(x)\mathrm{e}^{i2\omega x}\,dx$. *Use the rectangle with vertices at 0,* R*,* $R + i\omega$ *and* $i\omega$ *and the known integral* $\int_0^\infty \mathrm{e}^{-x^2}\,dx = \sqrt{\pi}/2$ *to show that* $I = \mathrm{e}^{-\omega^2}\sqrt{\pi}/2 + iB$. *Here* B *is the imaginary part and we are not concerned with its value.*

Solution: Let $g(z) = \mathrm{e}^{-z^2}\mathrm{e}^{i2z\omega}$. We go through the 4 sides of the rectangle one at a time.



On C_1, $z = x$, with x from 0 to R. So,

$$\lim_{R\to\infty}\int_{C_1} g(z)\,dz = \int_0^\infty \mathrm{e}^{-x^2}\mathrm{e}^{i2x\omega}\,dx = I.$$

On C_2, $z = R + iy$, with y from 0 to ω; $dz = i\,dy$; $z^2 = R^2 - y^2 + i2Ry$; $2iz\omega = 2iR\omega - 2y\omega$. So,

$$\left|\int_{C_2} g(z)\,dz\right| = \left|\int_0^\omega \mathrm{e}^{-R^2}\mathrm{e}^{y^2}\mathrm{e}^{-2iRy}\mathrm{e}^{2iR\omega}\mathrm{e}^{-2y\omega}\,i\,dy\right| \le \mathrm{e}^{-R^2}\int_0^\omega \mathrm{e}^{y^2}\mathrm{e}^{-2y\omega}\,dy.$$

Clearly this goes to 0 as R goes to ∞.

On C_3, $z = x + i\omega$, with x from 0 to R; $dz = dx$; $z^2 = x^2 - \omega^2 + 2ix\omega$; $2iz\omega = 2ix\omega - 2\omega^2$. So,

$$\int_{C_3} g(z)\,dz = \int_0^R \mathrm{e}^{-x^2}\mathrm{e}^{\omega^2}\mathrm{e}^{-2ix\omega}\mathrm{e}^{2ix\omega}\mathrm{e}^{-2\omega^2}\,dx = \mathrm{e}^{-\omega^2}\int_0^R \mathrm{e}^{-x^2}\,dx.$$

As $R \to \infty$, this integral goes to $\mathrm{e}^{-\omega^2}\dfrac{\sqrt{\pi}}{2}$.

On C_4, $z = iy$, with y from 0 to ω; $dz = i\,dy$; $z^2 = -y^2$; $2iz\omega = -2y\omega$. So,

$$\int_{C_4} g(z)\,dz = \int_0^\omega \mathrm{e}^{-y^2}\mathrm{e}^{-2y\omega}\,i\,dy.$$

This is pure imaginary, call it iB.

Since $g(z)$ is entire, we have $\int_{C_1+C_2-C_3-C_4} g(z)\,dz = 0$. So, using the limits above, we have

$$\lim_{R\to\infty}\int_{C_1+C_2-C_3-C_4} g(z)\,dz = I - \mathrm{e}^{-\omega^2}\frac{\sqrt{\pi}}{2} - iB.$$

This is what we needed to show!

(b) *Now use part (a) and symmetry to show that the Fourier transform* $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\omega x}\,dx = \sqrt{\pi}\mathrm{e}^{-\omega^2/4}$.

Solution: Since, $f(x) = \mathrm{e}^{-x^2}$ is an even function we have

$$\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\omega x}\,dx = \int_0^{\infty} f(x)(\mathrm{e}^{-i\omega x}+\mathrm{e}^{i\omega x})\,dx = \int_0^{\infty} f(x)2\cos(\omega x)\,dx = 2\,\mathrm{Re}\left(\int_0^{\infty} f(x)\mathrm{e}^{i\omega x}\,dx\right)$$

Part (a) implies $\int_0^{\infty} f(x)\mathrm{e}^{i\omega x}\,dx = \int_0^{\infty} f(x)\mathrm{e}^{2i(\omega/2)x}\,dx = \mathrm{e}^{-\omega^2/4}\frac{\sqrt{\pi}}{2} + iB$. So, we have that the Fourier transform

$$\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\omega x}\,dx = 2\,\mathrm{Re}\left(\int_0^{\infty} f(x)\mathrm{e}^{i\omega x}\,dx\right) = \mathrm{e}^{-\omega^2/4}\sqrt{\pi}. \quad \text{QED}$$

Problem Fun.2. (No points)

Compute $\int_0^{2\pi} (\cos\theta)^{2n}\,d\theta$. *For* $n = 1, 2, \ldots$. $\left(\textit{Answer: } \frac{2\pi \cdot (2n)!}{2^{2n}(n!)^2}.\right)$

Solution: Let

$$f(z) = \frac{1}{iz}\left(\frac{z + 1/z}{2}\right)^{2n} = \frac{(z^2+1)^{2n}}{i2^{2n}z^{2n+1}}.$$

We know that

$$\int_0^{2\pi} (\cos\theta)^{2n}\,d\theta = \int_{|z|=1} f(z)\,dz = 2\pi\,\mathrm{Res}(f, 0).$$

Expand $(z^2+1)^{2n}$ using the binomial theorem:

$$(z^2+1)^{2n} = \ldots + \frac{(2n)!}{n!n!}z^{2n} + \ldots$$

From this it's clear that $\mathrm{Res}(f, 0) = \frac{1}{i2^{2n}} \cdot \frac{(2n)!}{n!n!}$.

Thus, the integral in question is $\frac{2\pi \cdot (2n)!}{2^{2n}n!n!}$, as asserted.

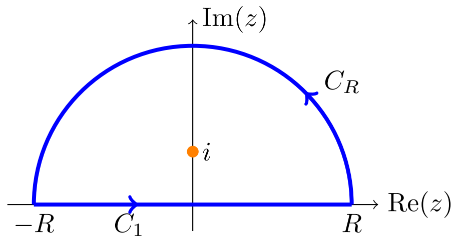
Problem Fun.3. (No points)

Compute p.v. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$.

Is this the same as the integral $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx$ *without the principal value?*

Solution: The integrand $f(x) = \frac{x^2}{(x^2+1)^2}$ has no singularities on the x-axis and is asymptotic to $1/x^2$, so the integral converges absolutely. This implies the integral is the same with or without the principal value.

Let I be the integral in question. Since $|f(z)|$ decays like $1/|z|^2$, we can use a semicircular contour.



The residue theorem implies $\displaystyle\int_{C_1+C_R} f(z)\,dz = 2\pi i \sum$ residues of f inside the contour. We examine each of the pieces in this equation.

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0. \qquad \text{(Theorem 9.1 in the Topic 9 notes.)}$$

$$\lim_{R\to\infty}\int_{C_1} f(z)\,dz = I. \qquad \text{(This is clear.)}$$

The only pole of f inside the contour is at $z = i$. This is a pole of order 2. Letting $g(z) = (z-i)^2 f(z) = \dfrac{z^2}{(z+i)^2}$, we have $\mathrm{Res}(f, i) = g'(i) = -\dfrac{i}{4}$..

Thus, $\boxed{I = \dfrac{\pi}{2}}$