18.04 Problem Set 8, Spring 2018

Calendar
Apr. 23-28: Reading: Topic 10 (Conformal maps)

Coming next

May 1-5: Applications

Problem 1. (15 points)
Let $z = x + iy$. Describe the image of each of the following regions under the mapping $w = \mathrm{e}^z$.

(a) The strip $0 < y < \pi$.

(b) The slanted strip between the lines $y = x$ and $y = x + 2\pi$.

(c) The half-strip $x > 0$, $0 < y < \pi$.

(d) The rectangle $1 < x < 2$, $0 < y < \pi$.

(e) The right half-plane $x > 0$.

Problem 2. (18 points)
(a) Find a fractional linear transformation that maps the right half-plane to the unit disk such that the origin is mapped to -1.

(b) A fixed point z of a transformation T is one where $T(z) = z$. Let T be a fractional linear transformation. Assume T is not the identity map. Show T has a most two fixed points.

(c) Let S be a circle and z_1 a point not on the circle. Show that there is exactly one point z_2 such that z_1 and z_2 are symmetric with respect to S.

(Hint: start by proving this for S a line.)

Problem 3. (20 points)
Suppose you want to find a function u harmonic on the right half-plane that takes the values $u(0, y) = y/(1 + y^2)$ on the imaginary axis. The first obvious guess is $u(z) = \mathrm{Im}(z/(1 - z^2))$. But this fails because $z/(1-z^2)$ has a singularity at $z = 1$. Find a valid u using the following steps.

So, forget about this guess and go back to only knowing that u is harmonic and $u(0, y) = y/(1 + y^2)$.

(a) Show that rotation by α is a fractional linear transformation which corresponds to the matrix $\begin{pmatrix} \mathrm{e}^{i\alpha/2} & 0 \\ 0 & \mathrm{e}^{-i\alpha/2} \end{pmatrix}$.

(This is not hard, it's just here in case you need it in part (b).)

(b) Find a fractional linear transformation that maps the right half-plane to the unit disk, so that u is transformed to a function ϕ with $\phi(\mathrm{e}^{i\theta}) = \sin(\theta)/2$.

Hint: make sure 1 is mapped to 0. If your transformation still doesn't transform u to the correct ϕ try composing with a rotation.

(c) Show that $\phi(w) = \dfrac{1}{2}\,\mathrm{Im}(w)$.

(d) Use the fractional linear transform to take ϕ back to u on the right half-plane.

Problem 4. (12 points)
(a) Show that the mapping $w = z + 1/z$ maps the circle $|z| = a$ $(a \neq 1)$ to the ellipse

$$\frac{u^2}{(a+1/a)^2} + \frac{v^2}{(a-1/a)^2} = 1.$$

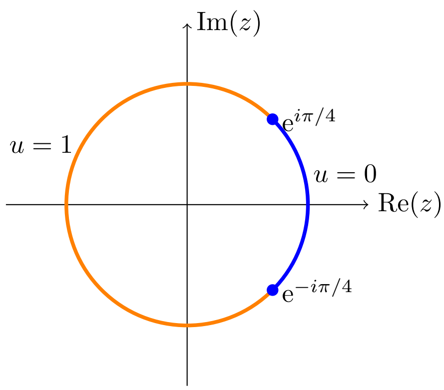
(b) Where does it map the circle $|z| = 1$?

(This problem will be helful when we look at Joukowsky transformations.)

Problem 5. (24 points)
(a) Find a harmonic function u on the upper half-plane that has the following boundary values.

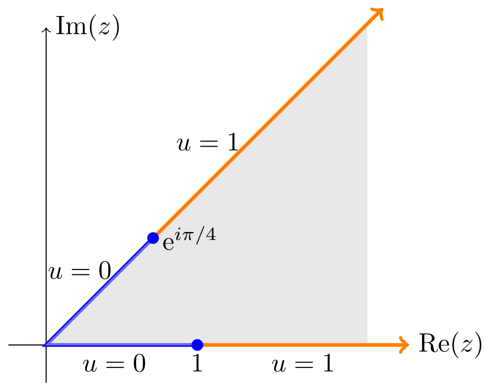
$$u(x,0) = \begin{cases} 1 & \text{for } x < -1 \\ 0 & \text{for } -1 < x < 1 \\ 1 & \text{for } 1 < x \end{cases}$$

(b) Find a harmonic function, $u(x, y)$, on the unit disk that boundary values indicated in the figure.



That is, $u(e^{i\theta}) = \begin{cases} 1 & \text{for } -\pi < \theta < \pi/4 \\ 0 & \text{for } -\pi/4 < \theta < \pi/4 \\ 1 & \text{for } \pi/4 < \theta < \pi \end{cases}$

(c) Find a harmonic function, $u(x, y)$, on the infinite wedge with angle $\pi/4$ shown. Such that u has the boundary values indicated in the figure.


Im(z)
$u = 1$
$e^{i\pi/4}$
$u = 0$
Re(z)
$u = 0$
1
$u = 1$