18.04 Problem Set 8, Spring 2018 Solutions

Problem 1. (15 points)
Let $z = x + iy$. Describe the image of each of the following regions under the mapping $w = \mathrm{e}^z$.

(a) *The strip $0 < y < \pi$.*

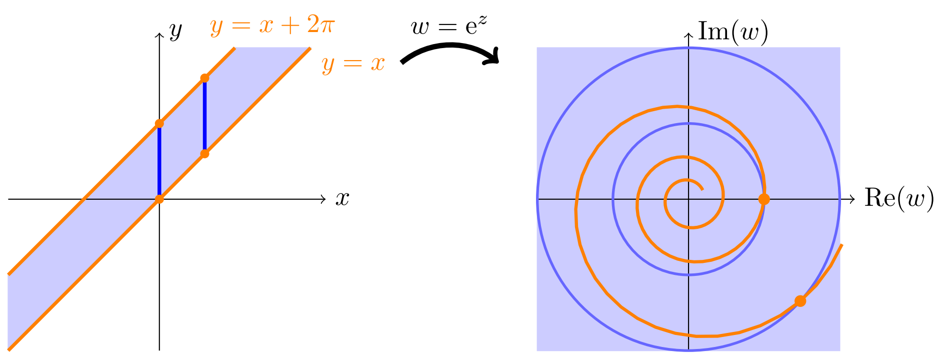
Solution: For this entire problem let's write $\mathrm{e}^z = r\mathrm{e}^{i\theta}$. Since $\mathrm{e}^z = \mathrm{e}^x\mathrm{e}^{iy}$, we have

$$r = \mathrm{e}^x, \qquad \theta = y$$

In the strip, e^x takes all values > 0. So, the image is every point with $0 < \theta < \pi$, i.e. $\boxed{\text{the upper half-plane.}}$

(b) *The slanted strip between the lines $y = x$ and $y = x + 2\pi$.*

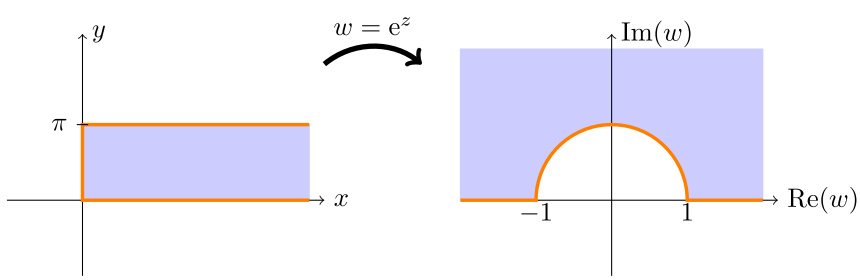
Solution: Both the boundary lines $y = x$ and $y = x + 2\pi$ map to the exponential spiral shown. Each of the two vertical segments on the left are of length 2π. So e^z maps this segment to a full circle minus the point $\mathrm{e}^{x+ix} = \mathrm{e}^{x+i(x+2\pi)}$. In this way, the slanted strip is mapped to the entire plane minus the spiral and minus the origin.



(c) *The half-strip $x > 0$, $0 < y < \pi$.*

Since $r = \mathrm{e}^x > 1$, the image is outside the unit circle. Since $0 < \theta = y < \pi$, the image is in the upper half plane.

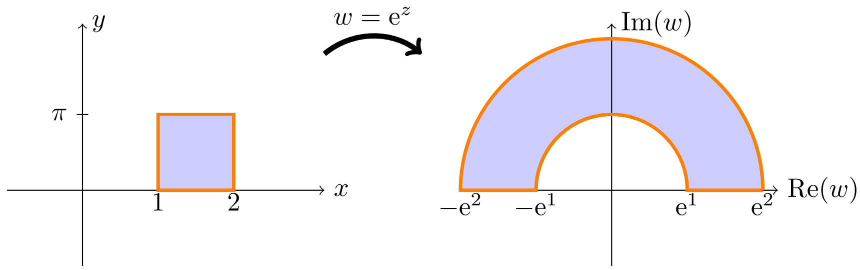
Answer: $\boxed{\text{The part of the upper half-plane outside the unit circle.}}$



(d) *The rectangle $1 < x < 2$, $0 < y < \pi$.*

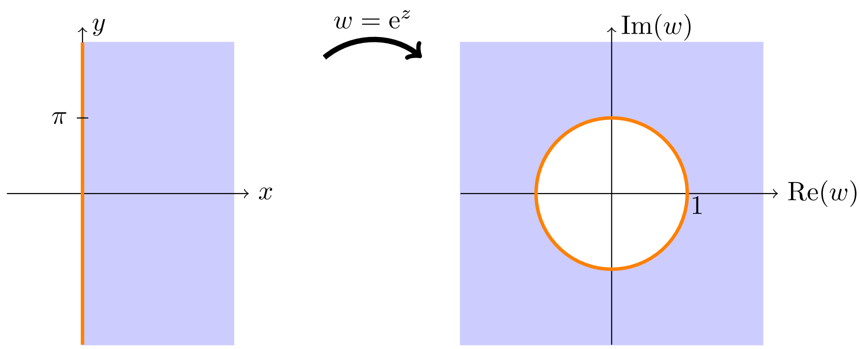
Solution: As before, since $0 < \theta = y < \pi$, the image is in the upper half-plane. The left and right edges of the rectangle map to the semicircles of radius e^1 and e^2 respectively. The top and bottom edges map to segments along the real axis

Answer: $\boxed{\text{The image is the half annulus shown.}}$



(e) *The right half-plane $x > 0$.*

Solution: Since $r = e^x > 0$ this maps to the exterior of the unit circle.



Problem 2. (18 points)

(a) *Find a fractional linear transformation that maps the right half-plane to the unit disk such that the origin is mapped to -1.*

Solution: $\boxed{T(z) = \dfrac{z-1}{z+1}}$. To verify this works we need to check three things.

1. $T(0) = -1$. This is obvious.

2. T maps the boundary of the right half-plane (y-axis) to the boundary of the unit disk (unit circle).

3. T maps the right half-plane to the inside of the disk.

To see (2): Take iy on the boundary of the half-plane. Then, $|T(iy)| = \dfrac{|iy-1|}{|iy+1|} = \dfrac{\sqrt{1+y^2}}{\sqrt{1+y^2}} = 1$. This shows $T(iy)$ is on the unit circle. Since we know fractional linear transformations map lines to lines or circles this shows the image of the y-axis must be the entire unit circle.

To see (3): (1) shows that one point in the right half-plane maps to the inside of the disk. This implies the entire half-plane must do so also.

(b) *A fixed point z of a transformation T is one where $T(z) = z$. Let T be a fractional linear transformation. Assume T is not the identity map. Show T has a most two fixed* *points.*

Let $T(z) = \dfrac{az+b}{cz+d}$. To find fixed points we solve the equation $T(z) = z$. A little algebra puts the equation in the form

$$\frac{az+b}{cz+d} = z \Leftrightarrow cz^2 + (d-a)z - b = 0.$$

There are a number of cases to consider. In all of them the number of fixed points is at most two.

If $c \neq 0$ then the quadratic equation has at most two solutions. (It might have only one if it has a repeated root.)

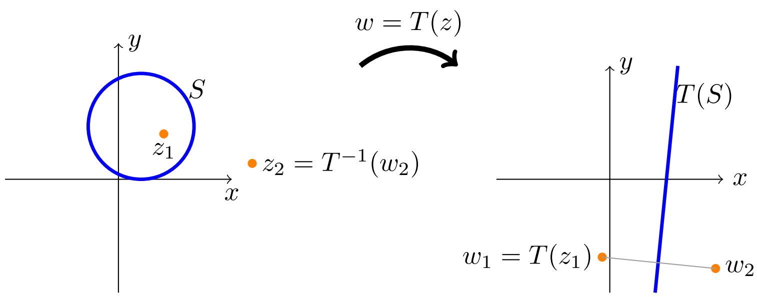
If $c = 0$ then the equation is linear and there is at most one solution. (There may be none.)

(c) *Let S be a circle and z_1 a point not on the circle. Show that there is exactly one point z_2 such that z_1 and z_2 are symmetric with respect to S.*

(Hint: start by proving this for S a line.)

This is in the topic 10 notes in section 10.3.2:

Let T be a fractional linear transformation that maps S to a line. We know that $w_1 = T(z_1)$ has a unique reflection w_2 in this line. Since T^{-1} preserves symmetry, z_1 and $z_2 = T^{-1}(w_2)$ are symmetric in S. Since w_2 is the unique point symmetric to w_1 the same is true for z_2 vis-a-vis z_1.



Problem 3. (20 points)
Suppose you want to find a function u harmonic on the right half-plane that takes the values $u(0,y) = y/(1+y^2)$ on the imaginary axis. The first obvious guess is $u(z) = \mathrm{Im}(z/(1-z^2))$. But this fails because $z/(1-z^2)$ has a singularity at $z = 1$. Find a valid u using the following steps.

So, forget about this guess and go back to only knowing that u is harmonic and $u(0,y) = y/(1+y^2)$.

(a) *Show that rotation by α is a fractional linear transformation which corresponds to the matrix $\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$.*

(This is not hard, it's just here in case you need it in part (b).)

Solution: $T(z) = \dfrac{e^{i\alpha/2}z}{e^{-i\alpha/2}} = e^{i\alpha}z$. This is indeed rotation by α.

(b) *Find a fractional linear transformation that maps the right half-plane to the unit disk, so that u is transformed to a function ϕ with $\phi(e^{i\theta}) = \sin(\theta)/2$.*

Hint: make sure 1 is mapped to 0. If your transformation still doesn't transform u to the correct ϕ try composing with a rotation.

Solution: Conveniently, the map found in problem 2a will work. Write $w = T(z)$, so z is the variable on the half-plane and w is the variable on the disk. We have the following ways of describing the ϕ.

$$u(z) = \phi \circ T(z) \qquad \text{or} \qquad \phi(w) = u \circ T^{-1}(w).$$

We have $T^{-1}(w) = \dfrac{w+1}{-w+1}$. We know that T^{-1} maps unit circle $e^{i\theta}$ to the imaginary axis iy. (Don't lose track of i multiplying the y!) so we have

$$iy = T^{-1}(e^{i\theta}) = \frac{e^{i\theta}+1}{-e^{i\theta}+1} \qquad \text{equivalently} \qquad y = \frac{1}{i} \cdot \frac{1+e^{i\theta}}{1-e^{i\theta}}$$

Now we can compute

$$\phi(e^{i\theta}) = u(0,y) = \frac{y}{1+y^2} = \frac{\frac{1}{i}\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)}{1-\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2} = \frac{1}{i} \cdot \frac{(1+e^{i\theta})(1-e^{i\theta})}{(1-e^{i\theta})^2-(1+e^{i\theta})^2}$$
$$= \frac{1}{i}\left(\frac{1-e^{i2\theta}}{-4e^{i\theta}}\right) = \frac{e^{-i\theta}-e^{i\theta}}{-4i} = \frac{1}{2} \cdot \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2}\sin(\theta).$$

This is exactly what was asserted in the problem.

(c) *Show that $\phi(w) = \dfrac{1}{2}\operatorname{Im}(w)$.*

Solution: The point is that we are looking for a function $\phi(w)$ which is
1. a harmonic function on the disk
2. has boundary value $\phi(e^{i\theta}) = \sin(\theta)/2$. That is, if $w = e^{i\theta}$ then we need to have $\phi(w) = \sin(\theta)/2$.

The function $\Phi(w) = w$ is analytic, so $\phi(w) = \dfrac{1}{2}\operatorname{Im}(w)$ is harmonic. On the boundary of the disk we have

$$\phi(e^{i\theta}) = \frac{1}{2}\operatorname{Im}(e^{i\theta}) = \frac{1}{2}\sin(\theta).$$

This is exactly what is required by requirement (2) above.

(d) *Use the fractional linear transform to take ϕ back to u on the right half-plane.*

Solution: The relationship $u(z) = \phi(T(z))$ gives

$$u(z) = \phi\left(\frac{z-1}{z+1}\right) = \frac{1}{2}\operatorname{Im}\left(\frac{z-1}{z+1}\right).$$

So, using $z = x+iy$

$$u(x,y) = \frac{1}{2}\operatorname{Im}\left(\frac{(x-1)+iy}{(x+1)+iy}\right) = \boxed{\frac{y}{(x+1)^2+y^2}}.$$

Problem 4. (12 points)
(a) *Show that the mapping $w = z + 1/z$ maps the circle $|z| = a$ $(a \neq 1)$ to the ellipse*

$$\frac{u^2}{(a+1/a)^2} + \frac{v^2}{(a-1/a)^2} = 1.$$

Solution: We'll use the notation $w = u + iv$. We'll write points on the circle $|z| = a$ as $ae^{i\theta}$. So,

$$\begin{aligned} w = z + \frac{1}{z} &= ae^{i\theta} + \frac{1}{ae^{i\theta}} \\ &= (a\cos(\theta) + ia\sin(\theta)) + \frac{\cos(\theta) - i\sin(\theta)}{a} \\ &= \left(a + \frac{1}{a}\right)\cos(\theta) + i\left(a - \frac{1}{a}\right)\sin(\theta). \end{aligned}$$

This shows $u = \left(a + \frac{1}{a}\right)\cos(\theta)$ and $\left(a - \frac{1}{a}\right)\sin(\theta)$. Now it is clear that

$$\frac{u^2}{(a+1/a)^2} + \frac{v^2}{(a-1/a)^2} = \cos^2(\theta) + \sin^2(\theta) = 1. \quad \text{QED}$$

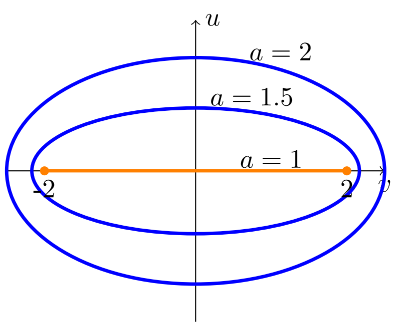
(b) *Where does it map the circle $|z| = 1$?*

(This problem will be helful when we look at Joukowsky transformations.)

Solution: If $z = e^{i\theta}$ then the computation above shows that

$$w = z + 1/z = 2\cos(\theta).$$

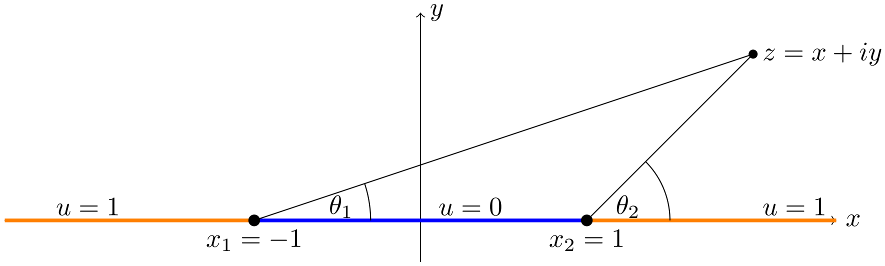
That is, w is real. So the unit circle is mapped to the $\boxed{\text{line segment } -2 \leq x \leq 2.}$ (As z goes once around the circle, w covers the segment twice.)



Problem 5. (24 points)
(a) *Find a harmonic function u on the upper half-plane that has the following boundary values.*

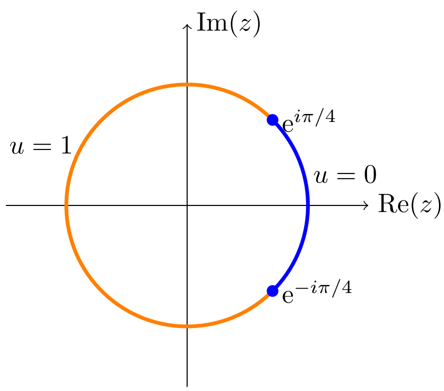
$$u(x,0) = \begin{cases} 1 & \text{for } x < -1 \\ 0 & \text{for } -1 < x < 1 \\ 1 & \text{for } 1 < x \end{cases}$$

Solution: $\boxed{u(z) = 1 - \frac{1}{\pi}\theta_2 + \frac{1}{\pi}\theta_1 = 1 - \frac{1}{\pi}\arg(z-1) + \frac{1}{\pi}\arg(z+1).}$



Here $\theta_1 = \arg(z+1)$, $\theta_2 = \arg(z-1)$, with θ_1 and θ_2 between $-\pi/2$ and $3\pi/2$. (We can choose any convenient branch cut that makes the argument analytic in the upper half-plane.)

(b) *Find a harmonic function, $u(x, y)$, on the unit disk that boundary values indicated in the figure.*



That is, $u(e^{i\theta}) = \begin{cases} 1 & \text{for } -\pi < \theta < \pi/4 \\ 0 & \text{for } -\pi/4 < \theta < \pi/4 \\ 1 & \text{for } \pi/4 < \theta < \pi \end{cases}$

Solution: Our strategy is to map the disk the the upper half-plane, solve the problem there and then translate back to the disk.

Method 1. Given our solution in part (a) we can find a fractional linear transformation T from the half-plane to the disk with

$$T(\infty) = -1, \quad T(-1) = e^{-i\pi/4}, \quad T(1) = e^{i\pi/4}.$$

With this map, our solution in part (a) on the half-plane would exactly correspond to the solution we seek. Finding such a T is straightforward. We start with $T(w) = \frac{-w+a}{w+b}$. This is chosen so that $T(\infty) = -1$. Plugging in $w = \pm 1$ we get the equations

$$T(-1) = \frac{1+a}{-1+b} = e^{-i\pi/4} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$
$$T(1) = \frac{-1+a}{1+b} = e^{i\pi/4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}.$$

These are easy to solve for

$$\boxed{a = \frac{1}{\sqrt{2}} + i\frac{(1+\sqrt{2})}{\sqrt{2}}, \qquad \text{and} \qquad b = i(1+\sqrt{2}).}$$

In any case, we will just use them as a and b.

Let's call our solution in part (a) $u_H(w)$. So

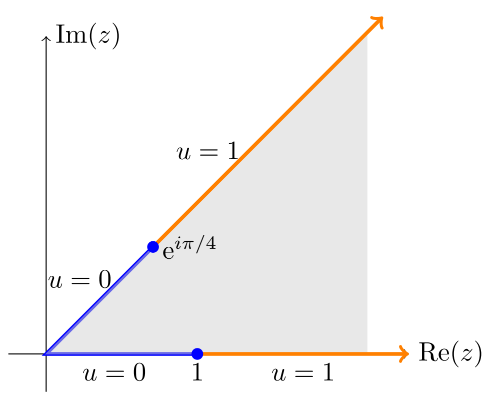
$$u_H(w) = 1 - \frac{1}{\pi}\theta_2 + \frac{1}{\pi}\theta_1 = \mathrm{Re}\left(1 - \frac{1}{\pi i}\log(w-1) + \frac{1}{\pi i}\log(w+1)\right).$$

The connection between u_H and the u we seek is $u(z) = u_H \circ T^{-1}(z)$. It's easy to compute that $w = T^{-1}(z) = \dfrac{bz-a}{-z+1}$, so

$$\boxed{u(z) = \mathrm{Re}\left(1 - \frac{1}{\pi i}\log\left(\frac{bz-a}{-z+1} - 1\right) + \frac{1}{\pi i}\log\left(\frac{bz-a}{-z+1} + 1\right)\right)}$$

Method 2. Here we just use our standard transform from the upper half-plane to the disk: $T(w) = \dfrac{z-i}{z+i}$. Let $w_1 = T^{-1}(\mathrm{e}^{i\pi/4})$, $w_2 = T^{-1}(\mathrm{e}^{-i\pi/4})$. Then we can transform the problem to the half-plane with jumps in the boundary value at w_1 and w_2. Solve this in the same manner as part (a) and transfer the solution back to the disk. We leave the calculations to you!

(c) *Find a harmonic function, $u(x,y)$, on the infinite wedge with angle $\pi/4$ shown. Such that u has the boundary values indicated in the figure.*



Solution: Call the wedge W and call the variable on the wedge w. The map $z = f(w) = w^4$ maps W to the upper half-plane, with $f(\mathrm{e}^{i\pi/4}) = -1$ and $f(1) = 1$.

So the boundary value problem in this part is transformed to exactly the problem in part (a). If we call the solution in part (a) $u_H(z)$, then the solution we seek is

$$\boxed{u(w) = u_H \circ f(w) = u_H(w^4) = 1 - \frac{1}{\pi}\arg(w^4-1) + \frac{1}{\pi}\arg(w^4+1).}$$