18.04 Problem Set 9, Spring 2018

Calendar
May 1-5: Reading: Topics 10, 11 (Conformal maps, Argument principle)

Coming next

May 8-12: Applications

Problem 1. (20 points)
(a) Start with a uniform flow and add a vortex at the point 2i. Give the complex potential for this flow. From that compute the stream function and sketch some streamlines. You can do this by hand or with a package like Matlab or Mathematica.

(b) Start with the same flow as part (a). Use the Milne-Thomson Theorem to make this flow go around the cylinder $|z| = 1$. Give the complex potential, the stream function and sketch some streamlines.

(c) Explain why the flows in both parts (a) and (b) look like uniform flow far from the origin.

Problem 2. (15 points)
Consider $f(z) = z^5 - 2z$.

(a) How many times does f wind the circle $|z| = 3$ around the origin? That is, let $\gamma(\theta)$ parametrize the circle. How many times does the curve $f \circ \gamma(\theta)$ wind around the origin.

(b) How many times does f wind the circle $|z| = 1$ around the origin?

(c) How many times does f wind the circle $|z| = 3$ around the point $z = -2$?

Problem 3. (10 points)
(a) Show that $f(z) = z^3 + 9z + 30$ has no roots in the disk $|z| < 2$.

(b) Show that $f(z) = z^6 + 4z^2 - 1$ has exactly two roots in the disk $|z| < 1$.

Problem 4. (7 points)
Suppose $f(z)$ is analytic on a region containing $|z| \leq 1$. Suppose also that $|f(z)| < 1$ on $|z| = 1$. Show that $f(z) - z$ has exactly one zero in the disk $|z| < 1$

Problem 5. (15 points)
In this problem we will consider linear systems with negative feedback, where the feedback gain is k. That is, if $G(s)$ is the open loop system function then the closed loop system function is $G_{CL}(s) = \dfrac{G(s)}{1 + kG(s)}$.

(a) Suppose a linear system has system function $G(s) = \dfrac{s+1}{(s-1)(s-2)}$. Let the feedback gain $k = 4$. Is the closed loop system stable? Do this analytically.

(b) For the system in part (a) draw the Nyquist plot, that is draw the curve $kG \circ \gamma$ where the curve γ is the y-axis. You can use any tool you want to do this. One suggestion is to use the

following applet: https://web.mit.edu/jorloff/www/jmoapplets/nyquist/nyquistCrit.html. (Use this version of the applet. I've modified the one we used in class to include feedback gain.)

How many times does this curve wind around -1? Explain how this is consistent with your answer to part (a).

Problem 6. (16 points)
Let $f(z) = (z + 1/z)/2$. Use a package like Matlab or Mathematic to draw $f \circ \gamma$ for the following curves.

Make sure that the axes both have the same scale, so that circles plot as circles and not ellipses. In Matlab the command `axis equal` after you plot will do this.

(a) The circle $|z| = 3/2$.

(b) The circle $|z + 1/2| = 3/2$.

(c) The circle $|e^{i\pi/4}z + 1/2| = 3/2$.

(d) The unit circle $|z| = 1$.

18.04 Complex Variables with Applications
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