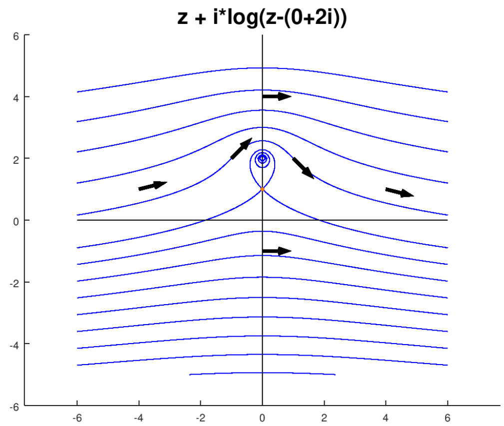
18.04 Problem Set 9, Spring 2018 Solutions

Problem 1. (20 points)
(a) *Start with a uniform flow and add a vortex at the point $2i$. Give the complex potential for this flow. From that compute the stream function and sketch some streamlines. You can do this by hand or with a package like Matlab or Mathematica.*

Solution: Here is the plot of some streamlines with a few velocity vectors added to show the direction. The orange dot on the negative y-axis is the stagnation point.



Uniform flow with a vortex at 0. Here, $U = 1$, $Q = 1$.

Uniform flow has complex potential $\Phi_U(z) = Uz$, where U is the direction.

A vortex at $z = 2i$ has complex potential $\Phi_V(z) = iQ\log(z - 2i)$, where Q is a measure of the circulation strength.

To avoid a lot of complicated arithmetic, let's set $U = 1$. So, the combined flow has complex potential $\boxed{\Phi(z) = z + iQ\log(z).}$

Let $z = x + iy = re^{i\theta}$, then

$\Phi(z) = (x+iy)+iQ(\log(|z-2i|)+i\arg(z-2i)) = (x-Q\arg(z-2i))+i(y+Q\log(|z-2i|)).$

So, the stream function is $\boxed{\psi = \mathrm{Im}(\Phi) = y + Q\log(|z-2i|)}$.

To plot some velocity vectors we used the fact the $\Phi' = u - iv$, where (u, v) is the velocity vector field. In this case,

$$\Phi'(z) = 1 + \frac{iQ}{z - 2i} = 1 + \frac{Q(y-2) + iQx}{|z-2i|^2}. \tag{1}$$

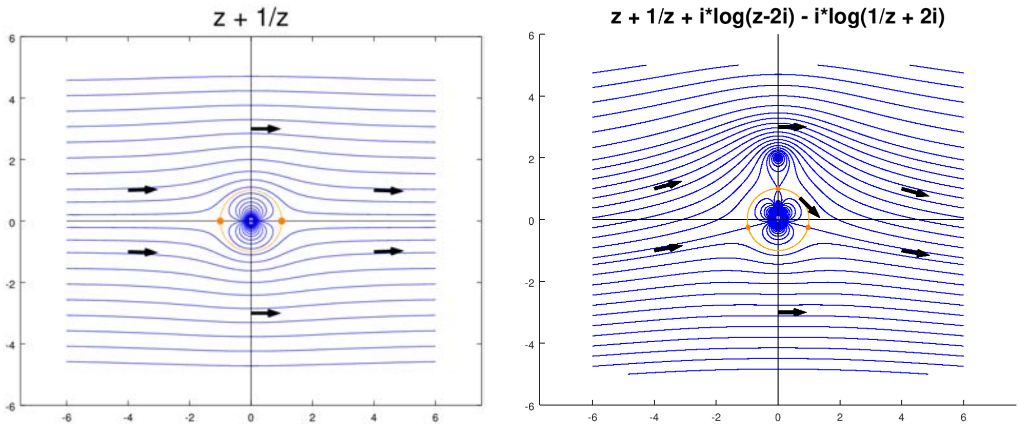
So, $(u, v) = \left(1 + \frac{Q(y-2)}{|z-2i|^2}, -\frac{Qx}{(|z-2i|^2}\right)$. Checking the velocity at $z = 1 + 2i$ we see the arrow points downward, so the circulation is clockwise around $z = 2i$.

Stagnation points are where $\Phi'(z) = 0$. In this case this there is only one stagnation point at $z = -Qi + 2i$. This shows that the stronger the circulation the farther out it pushes the stagnation point.

(b) *Start with the same flow as part (a). Use the Milne-Thomson Theorem to make this flow go around the cylinder $|z| = 1$. Give the complex potential, the stream function and sketch some streamlines.*

Solution: Below on the right is the flow asked for: uniform + vortex around a cylinder. As a bonus, on the left, we show uniform around a cylinder. Stagnation points are shown as orange dots. The cylinder is shown as an orange circle. We include the streamlines inside the cylinder because they look nice!

In the figures $U = 1$, $Q = 1$, $V = 2i$ and $R = 1$.



Left: uniform flow around a cylinder. Right: uniform plus vortex around a cylinder.

As before, uniform flow has $\Phi_U = Uz$. A vortex around $z = 2i$ has $\Phi_V(z) = iQ\log(z - 2i)$. So uniform plus vortex has complex potential

$$\Phi_{UV}(z) = Uz + iQ\log(z - 2i).$$

The Milne-Thomson circle theorem implies that this flow around a cylinder of radius R has complex potential

$$\Phi(z) = \Phi_{UV}(z) + \overline{\Phi_{UV}(R^2/\overline{z})} = Uz + \frac{\overline{U}R^2}{z} + iQ\log(z - 2i) - iQ\log(R^2/z + 2i).$$

The effect of U is just to scale and rotate the flow. Setting $U = 1$ makes the arithmetic a little simpler and doesn't affect the basic analysis, so let's do that. We have

$$\Phi(z) = z + \frac{R^2}{z} + iQ\log(z - 2i) - iQ\log(R^2/z + 2i).$$

So, for $z = x + iy$, the stream function is

$$\boxed{\psi = \mathrm{Im}(\Phi) = \left(y - \frac{R^2 y}{r^2} + Q\log(|z - 2i|) - Q\log(|R^2/z + 2i|)\right).}$$

Again we used Φ' to draw some arrows for the velocity field and find stagnation points.

$$\Phi'(z) = 1 - \frac{R^2}{z^2} + \frac{iQ}{z - 2i} + \frac{iQR^2}{z(R^2 + 2iz)}. \tag{2}$$

Stagnation points are where $\Phi'(z) = 0$. Since we end up with a fourth order polynomial, we used Octave's roots function to find the roots.

(c) *Explain why the flows in both parts (a) and (b) look like uniform flow far from the origin.*

Solution: In general, the velocity field (u, v) is related to $\Phi'(z)$ by $\Phi' = u - iv$.

In part (a) $\Phi'(z) = 1 + iQ/(z - 2i)$. As z goes to infinity $\Phi'(z) \approx 1$. This shows the vector field is approximately uniform for large z.

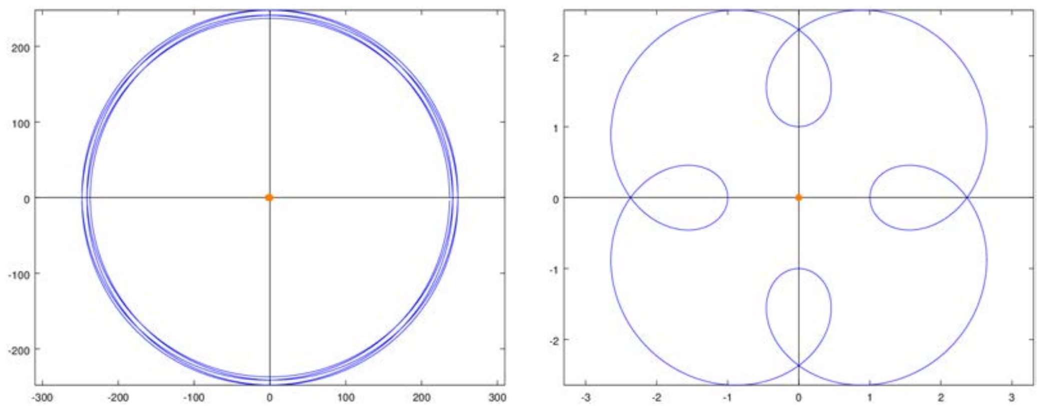
Part (b) is the same: The expression in Equation 2 for $\Phi'(z)$ is approximately 1 for large z.

Problem 2. (15 points)
Consider $f(z) = z^5 - 2z$.

(a) *How many times does f wind the circle $|z| = 3$ around the origin? That is, let $\gamma(\theta)$ parametrize the circle. How many times does the curve $f \circ \gamma(\theta)$ wind around the origin.*

Solution:

The figure on the left below show the plot of $f \circ \gamma$, where γ is the circle $|z| = 3$.



Left plot: parts (a) and (c). Right plot: part (b).

We will use the argument principle and Rouchés theorem to show that the winding number of $f \circ \gamma$ around 0 is $\boxed{\text{Ind}(f \circ \gamma, 0) = 5}$.

Using the notation from the class notes, the argument principle says that $\text{Ind}(f \circ \gamma, 0) = Z_{f,\gamma} - P_{f,\gamma}$. Since f is entire there are no poles and we have

$$\text{Ind}(f \circ \gamma, 0) = Z_{f,\gamma}$$

So, to show the winding number is 5, it suffices to show that f has exactly 5 zeros inside γ. For this, we split f into pieces: $f = g + z$, where $g(z) = z^5$ and $h(z) = -2z$. On $|z| = 3$ we have

$$|h| = 6 < 3^5 = |g|.$$

So Rouchés theorem implies that $Z_{g,\gamma} = Z_{g+h,\gamma}$. Clearly $Z_{g,\gamma} = 5$ This shows that $f = g + h$ has 5 zeros inside γ. QED.

(b) *How many times does f wind the circle $|z| = 1$ around the origin?*

Solution: The figure on the right above shows $f \circ \gamma$ where γ is the unit circle. The style of argument is identical to part (a).

Let $f = g + h$, where $g(z) = -2z$ and $h = z^5$. On $|z| = 1$, $|h| = 1 < 2 = |g|$. It is clear that g has exactly one zero inside γ. So, Rouchés theorem implies $Z_{f,\gamma} = Z_{g,\gamma} = 1$. That is, the winding number is 1.

(c) *How many times does f wind the circle $|z| = 3$ around the point $z = -2$?*

Solution: Let γ be the circle of radius 3 around the origin. Clearly, $\mathrm{Ind}(f \circ \gamma, -2) = \mathrm{Ind}(-2 + f \circ \gamma, 0)$. Now we apply Rouchés theorem to show that $f(z) - 2$ has 5 zeros inside γ. This will imply that $\boxed{\mathrm{Ind}(f \circ \gamma, -2) = 5.}$

Let $g(z) = z^5$. $|g| = 3^5$ on γ. Also g has 5 zeros inside γ.

Let $h(z) = -2z - 2$. $|h| \le 8$ on γ.

Since $|h| < |g|$ on γ, Rouchés theorem says that $f - 2 = g + h$ and g both have 5 zeros inside γ.

Problem 3. (10 points)
(a) *Show that $f(z) = z^3 + 9z + 30$ has no roots in the disk $|z| < 2$.*

We apply Rouchés theorem: let $g(z) = 9z + 30$ and $h(z) = z^3$. Both g and h are analytic and $f = g + h$.

On the curve $C : |z| = 2$ we have $|g| \ge |12|$ and $|h| = 8$, i.e. $|h| < |g|$. Since $|h|$ is strictly less than $|g|$ on C, Rouchés theorem implies that g and $f = g + h$ have the same number of zeros inside C. Since $g(z)$ clearly has no zeros inside the circle, neither does f.

(b) *Show that $f(z) = z^6 + 4z^2 - 1$ has exactly two roots in the disk $|z| < 1$.*

Let $g(z) = 4z^2 - 1$ and $h(z) = z^6$. On the unit circle we have

$$|h(z)| = 1 < 3 \le |g(z)|.$$

So, since g and h have no poles, Rouchés theorem says g and $f = g + h$ have the same number of zeros inside the unit circle. The roots of $g(z)$ are $\pm 1/2$, i.e. g has exactly two zeros in the disk, therefore, so does f.

Problem 4. (7 points)
Suppose $f(z)$ is analytic on a region containing $|z| \le 1$. Suppose also that $|f(z)| < 1$ on $|z| = 1$. Show that $f(z) - z$ has exactly one zero in the disk $|z| < 1$

Solution: We know that $g(z) = -z$ has exactly one zero in the unit disk. Since $|f| < 1 = |g|$ on the unit circle, we can apply Rouchés theorem to show that

$$1 = Z_{g,|z|=1} = Z_{f+g,|z|=1} = Z_{f-z,|z_=1} \qquad \text{QED.}$$

Problem 5. (15 points)
In this problem we will consider linear systems with negative feedback, where the feedback gain is k. That is, if $G(s)$ is the open loop system function then the closed loop system function is $G_{CL}(s) = \dfrac{G(s)}{1 + kG(s)}$.

(a) *Suppose a linear system has system function $G(s) = \dfrac{s+1}{(s-1)(s-2)}$. Let the feedback gain $k = 4$. Is the closed loop system stable? Do this analytically.*

Solution: We have

$$G_{CL} = \frac{G}{1+kG} = \frac{\frac{s+1}{(s-1)(s-2)}}{1+\frac{k(s+1)}{(s-1)(s-2)}} = \frac{s+1}{(s-1)(s-2)+k(s+1)}.$$

For $k = 4$ this becomes $G_{CL}(s) = \dfrac{s+1}{s^2+s+6}$. It's easy to compute that G_{CL} has poles $\dfrac{-1\pm\sqrt{-23}}{2}$. Since these are in the left half-plane, $\boxed{G_{CL} \text{ is stable.}}$

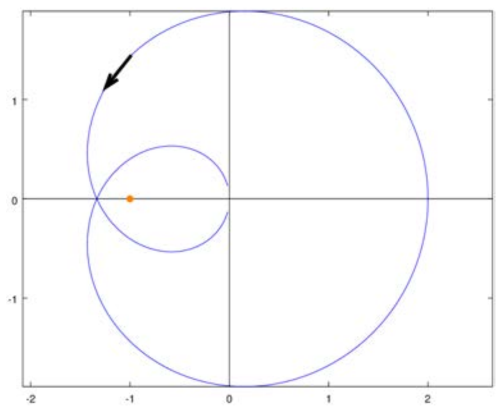
(b) *For the system in part (a) draw the Nyquist plot, that is draw the curve $kG\circ\gamma$ where the curve γ is the y-axis. You can use any tool you want to do this. One suggestion is to use the following applet:* `https://web.mit.edu/jorloff/www/jmoapplets/nyquist/nyquistCrit.html`. *(Use this version of the applet. I've modified the one we used in class to include feedback gain.)*

How many times does this curve wind around -1? Explain how this is consistent with your answer to part (a).

Solution: The plot below shows the winding number of $4G\circ\gamma$ around -1 is 2. This is consistent with part (a) because:

1. $G(s)$ has two poles in the right half-plane.

2. the Nyquist criterion for stability says that if G_{CL} is stable then the winding number of the Nyquist plot around -1 equals the number of poles in the right half-plane.



Problem 6. (16 points)
Let $f(z) = (z+1/z)/2$. Use a package like Matlab or Mathematic to draw $f\circ\gamma$ for the following curves.

Make sure that the axes both have the same scale, so that circles plot as circles and not ellipses. In Matlab the command `axis equal` *after you plot will do this.*

(a) *The circle $|z| = 3/2$.*

(b) *The circle $|z+1/2| = 3/2$.*

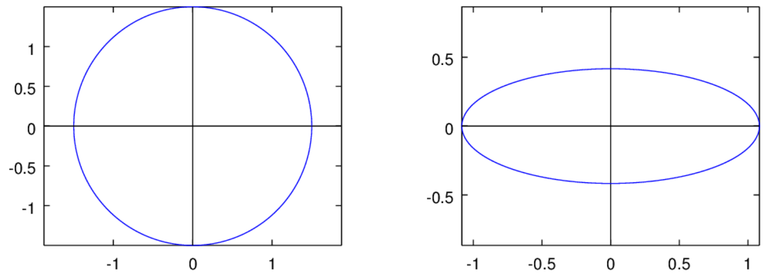
(c) *The circle $|e^{i\pi/4}z+1/2| = 3/2$.*

(d) *The unit circle $|z| = 1$.*

answers: We used the following Matlab (really Octave) code to do all the plots in this problem. For each plot all we had to do is change the values of `a` and `b`.

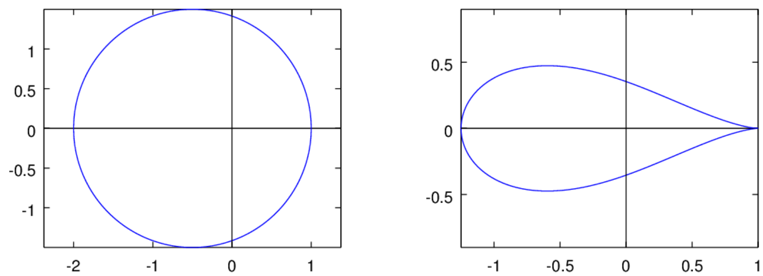
```
% Set the constants describing the circle

a = 1.5;
b = -0.5;

% Set theta from 0 to 2pi

th = [0:.01:2*pi];

% z is the circle being transformed
z = a*e.^(1i*th) + b;

% w is the transformed circle
w = 0.5*(z+1./z);

% Draw the plots
hold off   % This causes plot to erase the previous figure

% Subplot can be used to put more than one graph in a figure.  The command below
sets the figure to hold a 1 by 2 array of graphs.  The third argument sets the
current graph to draw in.

subplot(1,2,1)
plot(real(z),imag(z))   % Plot the circle

axis equal   % Set the scales on each axis to be the same.

hold on   % This allows us to overlay more plots on the current one.

% The follow bits of plotting draw axes through the origin.  There are other,
probably better, ways to do this.

plot(xlim(),[0,0],'k')
plot([0,0],ylim(),'k')

subplot(1,2,2) % Set the plot to draw in the second graph

plot(real(w),imag(w)) % Plot the transformed circle

axis equal
hold on
plot(xlim(),[0,0],'k')
plot([0,0],ylim(),'k')
hold off
```

(a) Here $a = 3/2$, $b = 0$. The image is an ellipse



(b) If $v = z + 1/2$ then $|v| = |z + 1/2| = 3/2$. So, $v = \dfrac{3}{2}\mathrm{e}^{i\theta}$, with $0 \le \theta \le 2\pi$.

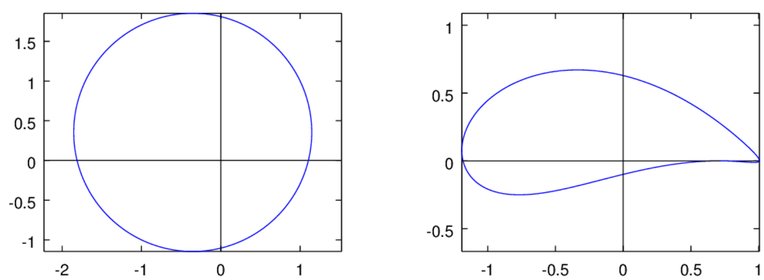
Solving for z: $z = v - 1/2 = \dfrac{3}{2}\mathrm{e}^{i\theta} - \dfrac{1}{2}$. So $a = 3/2$ and $b = -1/2$



(c) This is similar to part (b).

$$v = \mathrm{e}^{i\pi/4} z + \frac{1}{2} = \frac{3}{2}\mathrm{e}^{i\theta}, \quad \text{so,} \quad z = \mathrm{e}^{-i\pi/4}\left(\frac{3}{2}\mathrm{e}^{i\theta} - \frac{1}{2}\right)$$

That is, $a = \mathrm{e}^{-1i*\pi/4} * 1.5$ and $b = \mathrm{e}^{-1i*\pi/4} * (-0.5)$

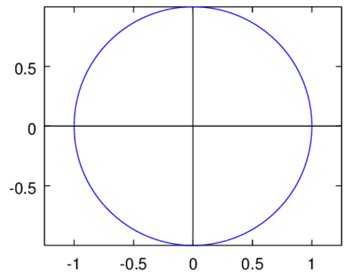


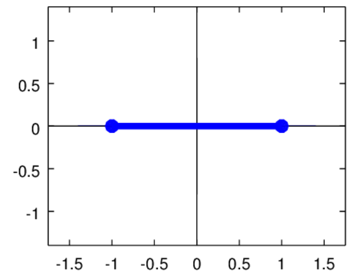
Notice how this looks like an airfoil. We could use a more complicated shaping functions to map the circle to other shapes. This map would map a flow around the cylinder to a flow around the new shape. For example, this could be used to study flow around a wing.

(d) Here $a = 1$ and $b = 0$. The result is a line segment on the real axis between -1 and 1. It's easy to see why this happens: on the unit circle $z = \mathrm{e}^{i\theta}$, so

$$\frac{z + 1/z}{2} = \frac{\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}}{2} = \cos(\theta).$$

This is real and stays between -1 and 1.


0.5
0
-0.5
-1
-0.5
0
0.5
1


1
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