

Class 2 in-class problems, 18.05, Spring 2022

Concept questions

Concept question 1. (What's the event?)

(Connecting words and set notation.)

Experiment: toss a coin 3 times.

Which of following equals the event "exactly two heads"?

$$A = \{THH, HTH, HHT, HHH\}$$

$$B = \{THH, HTH, HHT\}$$

$$C = \{HTH, THH\}$$

(1) A (2) B (3) C (4) B or C

Solution: : (2) B.

The event "exactly two heads" determines a *unique subset*, containing *all* outcomes that have exactly two heads.

Concept question 2. (Describe the event.)

(Connecting words and set notation.)

Experiment: toss a coin 3 times.

Which of the following describes the event $\{THH, HTH, HHT\}$?

(1) "exactly one head"

(2) "exactly one tail"

(3) "at most one tail"

(4) none of the above

Solution: (2) "exactly one tail"

Notice that the same event E may be described in words in multiple ways, e.g. "exactly 2 heads" or "exactly 1 tail".

Concept question 3. (Are they disjoint?)

(Connecting words and set notation.)

Experiment: toss a coin 3 times.

The events "exactly 2 heads" and "exactly 2 tails" are disjoint is.

(1) True (2) False

Solution: True: $\{THH, HTH, HHT\} \cap \{TTH, THT, HTT\} = \emptyset$.

Concept question 4. (Does A imply B?)

(Connecting words and set notation)

Consider two events: A and B.

Are the words "A implies B" equivalent to $A \subseteq B$?

(1) True (2) False

Solution: True.

For example: If you tossed “exactly 2 heads”, then you know you tossed “at least two heads”. This says exactly the same thing as

$$\{THH, HTH, HHT\} \subset \{THH, HTH, HHT, HHH\}.$$

Board problems

Problem 1. Poker hands

Deck of 52 cards

- 13 ranks: 2, 3, ..., 9, 10, J, Q, K, A
- 4 suits: ♡, ♠, ♦, ♣,

A poker hand consists of 5 cards

A one-pair hand consists of two cards having one rank and the remaining three cards having three other ranks

Example: $\{2♡, 2♠, 5♡, 8♣, K♦\}$

(a) How many different 5 card hands have exactly one pair?

Hint: practice with how many 2 card hands have exactly one pair.

Hint for hint: use the rule of product.

(b) What is the probability of getting a one pair poker hand?

(a) Solution: We need a systematic algorithm that produces every possible hand exactly once.

We can do this two ways: as combinations or as permutations. The keys are:

1. Be consistent.
2. Break the problem into a sequence of actions and use the rule of product.

Note, there are many ways to organize this. We will break it into very small steps in order to make the process clear.

Combinations approach

Consider a hand as a set of 5 cards, i.e. order doesn't matter.

Count the number of one-pair hands, by describing a procedure to make each one exactly once.

Action 1. *Choose* the rank of the pair: 13 different ranks, choosing 1, so $\binom{13}{1}$ ways to do this.

Action 2. *Choose* 2 cards from this rank: 4 cards in a rank, choosing 2, so $\binom{4}{2}$ ways to do this.

Action 3. *Choose* the 3 cards of different ranks: 12 remaining ranks, so $\binom{12}{3}$ ways to do this.

Action 4. *Choose* 1 card from each of these ranks: 4 cards in *each* rank so 4^3 ways to do this.

Answer (Using the rule of product.)

$$\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1098240$$

(b) To compute the probability we have to count all possible 5 card hands. We must stay consistent and count combinations.

To make a 5 card hand we *choose* 5 cards out of 52, so there are

$$\binom{52}{5} = 2598960$$

possible hands. Since each hand is equally probable, the probability of a one-pair hand is

$$1098240/2598960 = 0.42257.$$

We redo the problem using a **permutation approach**.

This approach is a little trickier. We include it to show that there is usually more than one way to count something.

(a) Count the number of one-pair hands, where we keep track of the order they are dealt.

Action 1. (This one is tricky.) *Choose* the positions in the hand that will hold the pair: 5 different positions, so $\binom{5}{2}$ ways to do this.

Action 2. Put a card in the first position of the pair: 52 cards, so 52 ways to do this.

Action 3. Put a card in the second position of the pair: since this has to match the first card, there are only 3 ways to do this.

Action 4. Put a card in the first open slot: this can't match the pair so there are 48 ways to do this.

Action 5. Put a card in the next open slot: this can't match the pair or the previous card, so there 44 ways to do this.

Action 6. Put a card in the last open slot: there are 40 ways to do this.

Answer: (Using the rule of product.)

$$\binom{5}{2} \cdot 52 \cdot 3 \cdot 48 \cdot 44 \cdot 40 = 131788800$$

ways to deal a one-pair hand where we keep track of order.

(b) There are

$${}_{52}P_5 = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311875200$$

five card hands where order is important. Thus, the probability of a one-pair hand is

$$131788800/311875200 = 0.42257.$$

(Both approaches give the same answer.)

Problem 2. (Inclusion-exclusion)

Supposes a class has 50 students: 20 male (M), 25 brown-eyed (B)

For a randomly chosen student, what is the range of possible values for $p = P(M \cup B)$?

- (a) $p \leq 0.4$*
- (b) $0.4 \leq p \leq 0.5$*
- (c) $0.4 \leq p \leq 0.9$*
- (d) $0.5 \leq p \leq 0.9$*
- (e) $0.5 \leq p$*

Solution: (d) $0.5 \leq p \leq 0.9$

Explanation:

The easy way to answer this is that $A \cup B$ has a minimum of 25 members (when all males are brown-eyed) and a maximum of 45 members (when no males have brown-eyes). So, the probability ranges from 0.5 to 0.9

Thinking about it in terms of the inclusion-exclusion principle we have

$$P(M \cup B) = P(M) + P(B) - P(M \cap B) = 0.9 - P(M \cap B).$$

So the maximum possible value of $P(M \cup B)$ happens if M and B are disjoint, so $P(M \cap B) = 0$. The minimum happens when $M \subset B$, so $P(M \cap B) = P(M) = 0.4$.

Problem 3. D20

Consider the following experiment. Roll a 20-sided die (D_{20}) 9 times. What is the probability that all 9 rolls are distinct.

(In class we could actually run this experiment many times with a real die and see how well the experimental data matched the theoretical probability. Later, we will learn how to simulate this experiment in R.)

For this experiment, how would you define the sample space, probability function, and event?

Compute the probability that all rolls (in one trial of 9 rolls) are distinct.

Solution: $\frac{20 \cdot 19 \cdots 13 \cdot 12}{20^9} = 0.119$.

Reasoning

For the sample space S we take all sequences of 9 numbers between 1 and 20. We find the size of S using the rule of product. There are 20 ways to choose the first number in the sequence, followed by 20 ways to choose the second, etc. Thus, $|S| = 20^9$. Of course, each of these outcomes is equally likely.

In our case, A is the event ‘all 9 numbers in the sequence are distinct’.

We can use the rule of product to compute $|A|$ as follows. There are 20 ways to choose the first number, then 19 ways to choose the second, etc. down to 12 ways to choose the ninth number. Thus, we have

$$|A| = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12$$

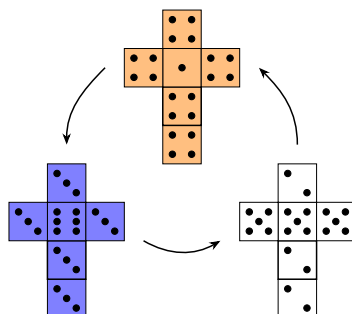
That is $|A| = {}_{20}P_9$.

Putting this all together

$$P(A) = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{20^9} \approx 0.119$$

Problem 4. Jon's Dice

Jon has three six-sided dice with unusual numbering.



A game consists of two players each choosing a die. They roll once and the highest number wins.

Which die would you choose?

1. Make probability tables for the blue and white dice.
2. Make a probability table for the product sample space of blue and white. That is, suppose you roll both dice and record the result as an ordered pair, (blue value, white value). List all the possible outcomes and their probabilities.
3. Use the table to compute the probability that blue beats white.
4. Pair up with another group. Have one group compare blue vs. orange and the other compare orange vs. white. Based on the three comparisons, rank the dice from best to worst.

Solution: Here are the probability tables we can use to compare the dice.

	Blue die		White die		Orange die	
Outcomes	3	6	2	5	1	4
Probability	5/6	1/6	3/6	3/6	1/6	5/6

- The 2×2 tables show pairs of dice.
- Each entry is the probability of seeing the pair of numbers corresponding to that entry.
- The color and letter gives the winning die for that pair of numbers. (We use black instead of white when the white die wins.)

		White		Orange	
		2	5	1	4
Blue	3	15/36 (B)	15/36 (W)	5/36 (B)	25/36 (O)
	6	3/36 (B)	3/36 (B)	1/36 (B)	5/36 (B)
Orange	1	3/36 (W)	3/36 (W)		
	4	15/36 (O)	15/36 (W)		

The three comparisons are:

$$\begin{aligned} P(\text{blue beats white}) &= 21/36 = 7/12 \\ P(\text{white beats orange}) &= 21/36 = 7/12 \\ P(\text{orange beats blue}) &= 25/36 \end{aligned}$$

Thus: blue is better than white is better than orange is better than blue.

There is no best die: the property of being ‘better than’ is not transitive.

Problem 5. Lucky Lucy

This is doable, but challenging problem. You should be able to compute $P(A)$ and $P(B)$. It takes a bit of algebraic cleverness to decide if one is always bigger than the other.

Lucky Lucy has a coin that you’re quite sure is not fair.

- They will flip the coin twice
- Let A be the event the tosses are the same, i.e. $\{HH, TT\}$
- Let B be the event the tosses are the different, i.e. $\{HT, TH\}$

Let p be the probability of heads. Compute and compare $P(A)$ and $P(B)$.

(If you don’t see the symbolic algebra try $p = 0.2$, $p=0.5$)

Solution: We’re given $P(H) = p$. It’s probably easiest to introduce a new letter: let $P(T) = q$. Of course, $q = 1 - p$, or $p + q = 1$.

We have the following table for two tosses

Outcomes	HH	TT	HT	TH
Probability	p^2	q^2	pq	qp

So, $P(A) = p^2 + q^2$ and $P(B) = 2pq$.

Here’s the trick: Since the coin is unfair, $p \neq q$. Thus $(p - q)^2 > 0$. Now expand this out:

$$(p - q)^2 > 0 \Rightarrow p^2 - 2pq + q^2 > 0 \Rightarrow p^2 + q^2 > 2pq.$$

The last inequality shows, as long as $p \neq q$, $P(A) > P(B)$. That is, Lucy is more likely to have both tosses the same.

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18.05 Introduction to Probability and Statistics

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