Class 3 in-class problems, 18.05, Spring 2022

Concept questions

Concept question 1.
Toss a coin 4 times. Let $A$ = ‘at least three heads’ and $B$ = ‘first toss is tails’.
1. What is $P(A|B)$?
   (a) 1/16 (b) 1/8 (c) 1/4 (d) 1/5
2. What is $P(B|A)$?
   (a) 1/16 (b) 1/8 (c) 1/4 (d) 1/5

Solution: 1. (b) 1/8. 2. (d) 1/5.

Counting we find $|A| = 5$, $|B| = 8$ and $|A \cap B| = 1$. Since all sequences are equally likely

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|} = 1/8. \quad P(B|A) = \frac{|B \cap A|}{|A|} = 1/5.$$

Concept question 2. Trees 1.

1. The probability $x$ represents
   (a) $P(A_1)$ (b) $P(A_1|B_2)$ (c) $P(B_2|A_1)$ (d) $P(C_1|B_2 \cap A_1)$.

Solution: (a) $P(A_1)$.

Concept question 3. Trees 2.

2. The probability $y$ represents
   (a) $P(B_2)$ (b) $P(A_1|B_2)$ (c) $P(B_2|A_1)$ (d) $P(C_1|B_2 \cap A_1)$.

Solution: (c) $P(B_2|A_1)$.

Concept question 4. Trees 3.

3. The probability $z$ represents
   (a) $P(C_1)$ (b) $P(B_2|C_1)$ (c) $P(C_1|B_2)$ (d) $P(C_1|B_2 \cap A_1)$.

Solution: (d) $P(C_1|B_2 \cap A_1)$.

Concept question 5. Trees 4.

4. The circled node represents the event
   (a) $C_1$ (b) $B_2 \cap C_1$ (c) $A_1 \cap B_2 \cap C_1$ (d) $C_1|B_2 \cap A_1$.

Solution: (c) $A_1 \cap B_2 \cap C_1$.
In class examples

Class example 1.

- Organize computations
- Compute total probability
- Compute Bayes’ formula

Example. Game: 5 orange and 2 blue balls in an urn. A random ball is selected and replaced by a ball of the other color; then a second ball is drawn.

1. What is the probability the second ball is orange?
2. What is the probability the first ball was orange given the second ball was orange?

\[ P(O_2) = \frac{5}{7} \cdot \frac{4}{7} + \frac{2}{7} \cdot \frac{6}{7} = \frac{32}{49} \]

Solution: 1. Let \( O_1 \) be the event the first ball is orange. Likewise for \( O_2, B_1, B_2 \). The law of total probability gives

\[ P(O_2) = \frac{5}{7} \cdot \frac{4}{7} + \frac{2}{7} \cdot \frac{6}{7} = \frac{32}{49} \]

2. Bayes’ rule gives

\[ P(O_1|O_2) = \frac{P(O_1 \cap O_2)}{P(O_2)} = \frac{20/49}{32/49} = \frac{20}{32} \]

Board questions

Problem 1. Monty Hall

- One door hides a car, two hide goats.
- The contestant chooses any door.
- Monty always opens a different door with a goat. (He can do this because he knows where the car is.)
- The contestant is then allowed to switch doors if they want.

What is the best strategy for winning a car?

(a) Switch    (b) Don’t switch    (c) It doesn’t matter
Organize the Monty Hall problem into a tree and compute the probability of winning if you always switch.

*Hint* first break the game into a sequence of actions.

**Solution:** Let \( P_{\text{switch}} \) be the probability function when the contestant uses the switching strategy. Let \( C \) represent a car and \( G \) a goat.

We will see that \( P_{\text{switch}}(C) = \frac{2}{3} \)

One way to show this is with a tree representing the switching strategy: First the contestant chooses a door, (then Monty shows a goat), then the contestant switches doors.

The (total) probability of \( C \) is \( P_{\text{switch}}(C) = \frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1 = \frac{2}{3} \).

Writing this in symbols using the law of total probability: Here \( G_1 \) represents a goat is behind the original door chosen and \( G_2 \) means a goat is behind the door that is switched to. Likewise \( C_1 \) and \( C_2 \).

\[
P_{\text{switch}}(C_2) = P_{\text{switch}}(C_2|C_1)P(C_1) + P_{\text{switch}}(C_2|G_1)P(G_1) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}.
\]

**Problem 2. Independence**

Roll two dice and consider the following events

- \( A = \) ‘first die is 3’
- \( B = \) ‘sum is 6’
- \( C = \) ‘sum is 7’

\( A \) is independent of

- (a) \( B \) and \( C \)
- (b) \( B \) alone
- (c) \( C \) alone
- (d) Neither \( B \) or \( C \).

**Solution:** (c). (Explanation below)

\( P(A) = 1/6, P(A|B) = 1/5. \) Not equal, so not independent.

\( P(A) = 1/6, P(A|C) = 1/6. \) Equal, so independent.
Notice that knowing $B$, removes 6 as a possibility for the first die and makes $A$ more probable. So, knowing $B$ occurred changes the probability of $A$.

But, knowing $C$ does not change the probabilities for the possible values of the first roll; they are still $1/6$ for each value. In particular, knowing $C$ occurred does not change the probability of $A$.

We could also have done this problem by showing

\[ P(B|A) \neq P(B) \text{ or } P(A \cap B) \neq P(A)P(B). \]

**Problem 3. Evil Squirrels**

*Of the one million squirrels on MIT’s campus most are good-natured. But one hundred of them are pure evil! An enterprising student in Course 6 develops an “Evil Squirrel Alarm” which they offer to sell to MIT for a passing grade. MIT decides to test the reliability of the alarm by conducting trials.*

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- When presented with an evil squirrel, the alarm goes off 99% of the time.
- When presented with a good-natured squirrel, the alarm goes off 1% of the time.

(a) If a squirrel sets off the alarm, what is the probability that it is evil?

(b) Should MIT co-opt the patent rights and employ the system?

**One solution**

(This is a base rate fallacy problem)

We are given:

\[ P(\text{nice}) = 0.9999, P(\text{evil}) = 0.0001 \text{ (base rate)} \]

\[ P(\text{alarm} | \text{nice}) = 0.01, P(\text{alarm} | \text{evil}) = 0.99 \]
\[ P(\text{evil} \mid \text{alarm}) = \frac{P(\text{alarm} \mid \text{evil}) P(\text{evil})}{P(\text{alarm})} = \frac{P(\text{alarm} \mid \text{evil}) P(\text{evil})}{P(\text{alarm} \mid \text{evil}) P(\text{evil}) + P(\text{alarm} \mid \text{nice}) P(\text{nice})} = \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.01)(0.9999)} \approx 0.01 \]

Summary:

Probability a random test is correct = 0.99
Probability a positive test is correct \approx 0.01

These probabilities are not the same!

Alternative method of calculation:

<table>
<thead>
<tr>
<th></th>
<th>Evil</th>
<th>Nice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alarm</td>
<td>99</td>
<td>9999</td>
</tr>
<tr>
<td>No alarm</td>
<td>1</td>
<td>989901</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>999900</td>
</tr>
</tbody>
</table>

Solution: (a) This is the same solution as above, but in a more compact notation. Let \( E \) be the event that a squirrel is evil. Let \( A \) be the event that the alarm goes off. By Bayes’ Theorem, we have:

\[ P(E \mid A) = \frac{P(A \mid E) P(E)}{P(A \mid E) P(E) + P(A \mid E^c) P(E^c)} = \frac{0.99 \cdot \frac{100}{1000000}}{(0.99) \cdot \left( \frac{100}{1000000} \right) + (0.01) \cdot \left( \frac{999900}{1000000} \right)} \approx 0.01. \]

(b) No. The alarm would be more trouble than its worth, since for every true positive there are about 99 false positives.

Problem 4. Dice Game

1. The Randomizer holds the 6-sided die in one fist and the 8-sided die in the other.
2. The Roller selects one of the Randomizer’s fists and covertly takes the die.
3. The Roller rolls the die in secret and reports the result to the table.

Given the reported number, what is the probability that the 6-sided die was chosen? (Find the probability for each possible reported number.)

Solution: If the number rolled is 1-6 then \( P(\text{six-sided}) = 4/7. \)
If the number rolled is 7 or 8 then \( P(\text{six-sided}) = 0 \).

This is a Bayes’ formula problem. For concreteness let’s suppose the roll was a 4. What we want to compute is \( P(\text{6-sided}|\text{roll 4}) \). But, what is easy to compute is \( P(\text{roll 4}|\text{6-sided}) \).

Bayes’ formula says

\[
P(\text{6-sided}|\text{roll 4}) = \frac{P(\text{roll 4}|\text{6-sided})P(\text{6-sided})}{P(4)} = \frac{(1/6)(1/2)}{(1/6)(1/2) + (1/8)(1/2)} = 4/7.
\]

The denominator is computed using the law of total probability:

\[
P(4) = P(4|\text{6-sided})P(\text{6-sided}) + P(4|\text{8-sided})P(\text{8-sided}) = \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{2}.
\]

Note that any roll of 1,2,...,6 would give the same result. A roll of 7 (or 8) would give clearly give probability 0. This is seen in Bayes’ formula because the term \( P(\text{roll 7}|\text{6-sided}) = 0 \).