

# Gallery of Continuous Random Variables

## Class 5, 18.05

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## 1 Learning Goals

1. Be able to give examples of what uniform, exponential and normal distributions are used to model.
2. Be able to give the range and pdf's of uniform, exponential and normal distributions.

## 2 Introduction

Here we introduce a few fundamental continuous distributions. These will play important roles in the statistics part of the class. For each distribution, we give the range, the pdf, the cdf, and a short description of situations that it models. These distributions all depend on parameters, which we specify.

As you look through each distribution do not try to memorize all the details; you can always look those up. Rather, focus on the shape of each distribution and what it models.

Although it comes towards the end, we call your attention to the normal distribution. It is easily the most important distribution defined here.

### 2.1 Parametrized distributions

When we studied discrete random variables we learned, for example, about the Bernoulli( $p$ ) distribution. The probability  $p$  used to define the distribution is called a **parameter** and Bernoulli( $p$ ) is called a **parametrized distribution**. For example, tosses of fair coin follow a Bernoulli distribution where the parameter  $p = 0.5$ . When we study statistics one of the key questions will be to estimate the parameters of a distribution. For example, if I have a coin that may or may not be fair then I know it follows a Bernoulli( $p$ ) distribution, but I don't know the value of the parameter  $p$ . I might run experiments and use the data to estimate the value of  $p$ .

As another example, the binomial distribution Binomial( $n, p$ ) depends on two parameters  $n$  and  $p$ .

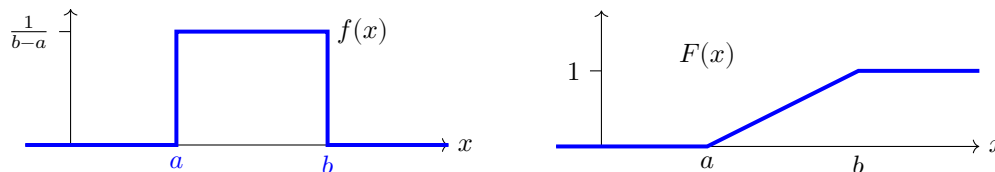
In the following sections we will look at specific parametrized continuous distributions. The applet <https://mathlets.org/mathlets/probability-distributions/> allows you to visualize the pdf and cdf of these distributions and to dynamically change the parameters.

## 3 Uniform distribution

1. Parameters:  $a, b$ .
2. Range:  $[a, b]$ .

3. Notation:  $\text{uniform}(a, b)$  or  $U(a, b)$ .
4. Probability density function (pdf):  $f(x) = \frac{1}{b-a}$  for  $a \leq x \leq b$ .
5. Cumulative distribution function (cdf):  $F(x) = (x-a)/(b-a)$  for  $a \leq x \leq b$ .
6. Models: Situations where all outcomes in the range have equal probability (more precisely all outcomes have the same probability density).

Graphs:



pdf and cdf for  $\text{uniform}(a,b)$  distribution.

**Example 1. 1.** Suppose we have a tape measure with markings at each millimeter. If we measure (to the nearest marking) the length of items that are roughly a meter long, the rounding error will be **uniformly distributed** between -0.5 and 0.5 millimeters.

**2.** Many board games use spinning arrows (spinners) to introduce randomness. When spun, the arrow stops at an angle that is uniformly distributed between 0 and  $2\pi$  radians.

**3.** In most pseudo-random number generators, the basic generator simulates a uniform distribution and all other distributions are constructed by transforming the basic generator.

## 4 Exponential distribution

1. Parameter:  $\lambda$ .
2. Range:  $[0, \infty)$ .
3. Notation:  $\text{exponential}(\lambda)$  or  $\exp(\lambda)$ .
4. Probability density function (pdf):  $f(x) = \lambda e^{-\lambda x}$  for  $0 \leq x$ .
5. Cumulative distribution function (cdf): (This is an easy integral.)

$$F(x) = 1 - e^{-\lambda x} \text{ for } x \geq 0$$

6. **Right tail distribution:**  $P(X > x) = 1 - F(x) = e^{-\lambda x}$ . (Note: this is defined as  $P(X > x)$ , i.e. that  $X$  is to the right of  $x$  on the number line.)

7. Models: The waiting time for a continuous process to change state.

**Example 2.** If I step out to 77 Mass Ave after class and wait for the next taxi, my waiting time in minutes is exponentially distributed. We will see that in this case  $\lambda$  is given by  $1/(\text{average number of taxis that pass per minute})$ .

**Example 3.** The exponential distribution models the waiting time until an unstable isotope undergoes nuclear decay. In this case, the value of  $\lambda$  is related to the half-life of the isotope.

**Memorylessness:** There are other distributions that also model waiting times, but the exponential distribution has the additional property that it is memoryless. Here's what this means in the context of Example 2: suppose that the probability that a taxi arrives within the first five minutes is  $p$ . If I wait five minutes and, in this case, no taxi arrives, then the probability that a taxi arrives within the next five minutes is still  $p$ . That is, my previous wait of 5 minutes has no impact on the length of my future wait!

By contrast, suppose I were to instead go to Kendall Square subway station and wait for the next inbound train. Since the trains are coordinated to follow a schedule (e.g., roughly 12 minutes between trains), if I wait five minutes without seeing a train then there is a far greater probability that a train will arrive in the next five minutes. In particular, waiting time for the subway is not memoryless, and a better model would be the uniform distribution on the range  $[0,12]$ .

The memorylessness of the exponential distribution is analogous to the memorylessness of the (discrete) geometric distribution, where having flipped 5 tails in a row gives no information about the next 5 flips. Indeed, the exponential distribution is precisely the continuous counterpart of the geometric distribution, which models the waiting time for a discrete process to change state. More formally, memoryless means that the probability of waiting  $t$  more minutes is independent of the amount of time already waited. In symbols,

$$P(X > s + t | X > s) = P(X > t).$$

**Proof of memorylessness:** We know that

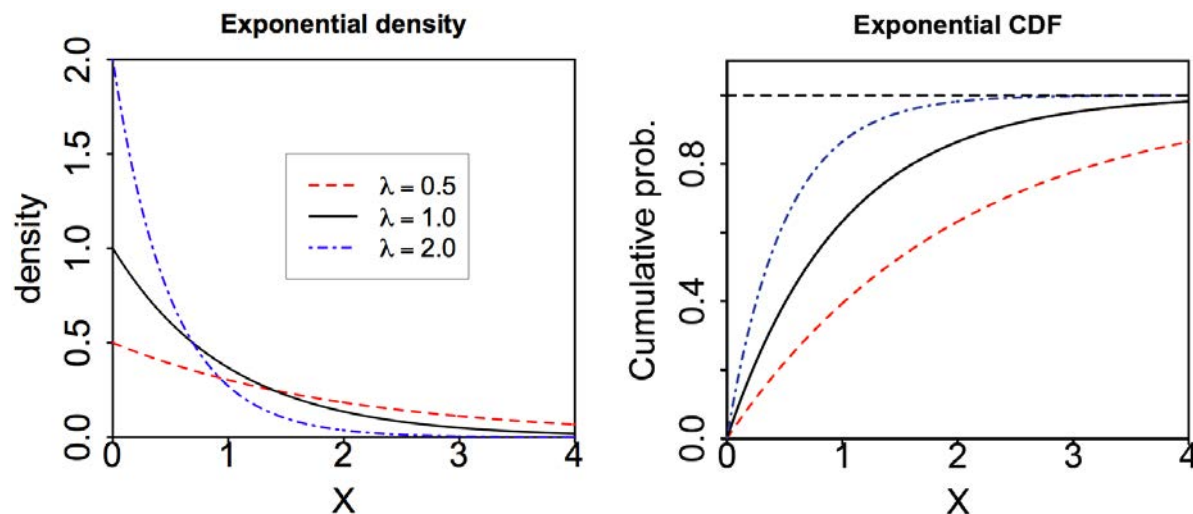
$$(X > s + t) \cap (X > s) = (X > s + t),$$

since the event 'waited at least  $s$  minutes' contains the event 'waited at least  $s + t$  minutes'. Therefore the formula for conditional probability gives

$$P(X > s + t | X > s) = \frac{P(X > s + t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t).$$

The probability  $P(X > s + t) = e^{-\lambda(s+t)}$  is the formula for the right tail probability given above.

Graphs:



## 5 Normal distribution

In 1809, [Carl Friedrich Gauss](#) published a monograph introducing several notions that have become fundamental to statistics: the normal distribution, maximum likelihood estimation, and the method of least squares (we will cover all three in this course). For this reason, the [normal distribution](#) is also called the [Gaussian distribution](#), and it is by far the most important continuous distribution.

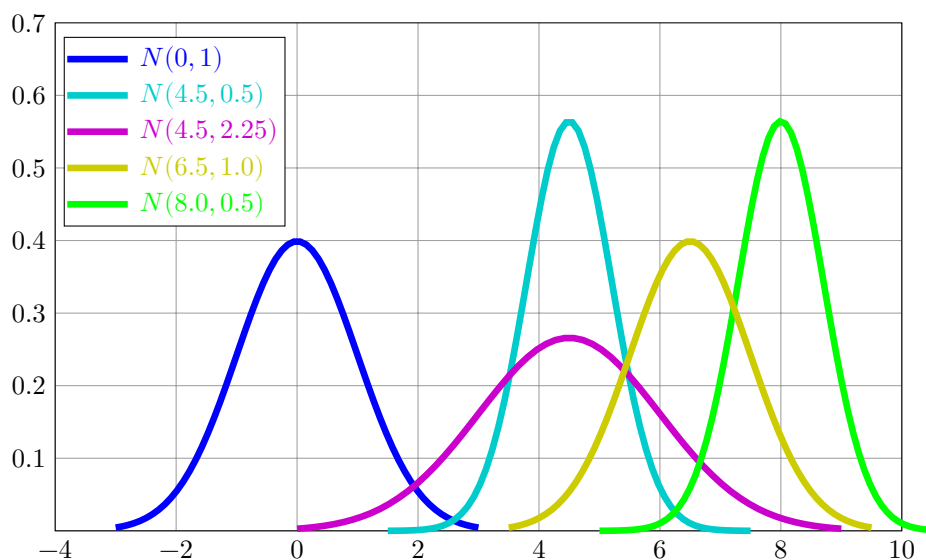
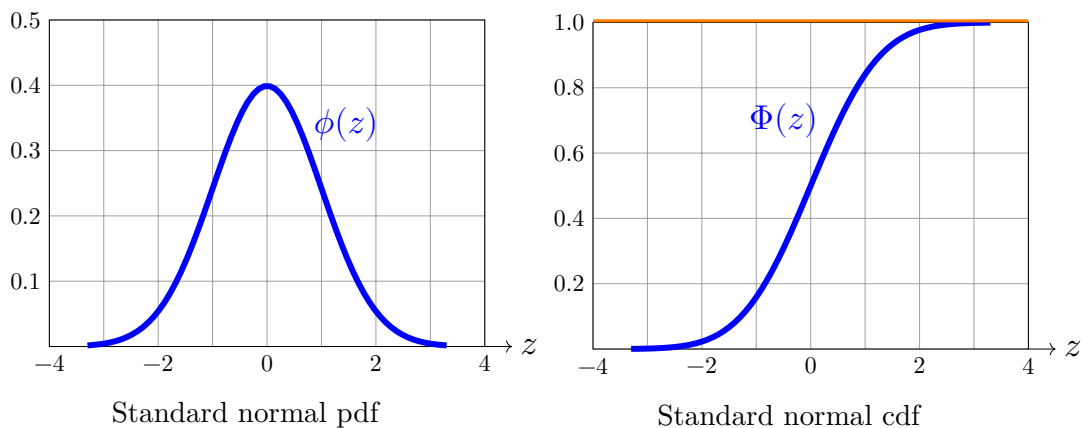
1. Parameters:  $\mu, \sigma$ .
2. Range:  $(-\infty, \infty)$ .
3. Notation:  $\text{Normal}(\mu, \sigma^2)$  or  $N(\mu, \sigma^2)$ .
4. Probability density function (pdf):  $f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$ .
5. Cumulative distribution function (cdf):  $F(x)$  has no formula, so use tables or software such as `pnorm` in R to compute  $F(x)$ .
6. Models: Measurement error, intelligence/ability, height, averages of lots of data.

The [standard normal distribution](#)  $N(0, 1)$  has mean 0 and variance 1. We reserve  $Z$  for a [standard normal random variable](#),  $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$  for the standard normal density, and  $\Phi(z)$  for the standard normal distribution.

**Note:** we will define mean and variance for continuous random variables next time. They have the same interpretations as in the discrete case. As you might guess, the normal distribution  $N(\mu, \sigma^2)$  has mean  $\mu$ , variance  $\sigma^2$ , and standard deviation  $\sigma$ .

Here are some graphs of normal distributions. Note that they are shaped like a [bell curve](#). Note also that as  $\sigma$  increases they become more spread out.

The **bell curve**: First we show the standard normal probability density and cumulative distribution functions. Below that is a selection of normal densities. Notice that the graph is centered on the mean and the bigger the variance the more spread out the curve.



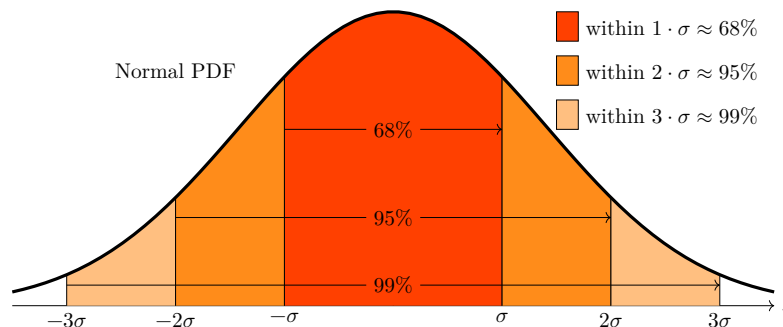
**Notation note.** In the figure above we use our notation  $N(\mu, \sigma^2)$ . So, for example,  $N(8, 0.5)$  has variance 0.5 and standard deviation  $\sigma = \sqrt{0.5} \approx 0.7071$ .

## 5.1 Normal probabilities

To make approximations it is useful to remember the following **rule of thumb** for three approximate probabilities from the standard normal distribution:

$$P(-1 \leq Z \leq 1) \approx 0.68, \quad P(-2 \leq Z \leq 2) \approx 0.95, \quad P(-3 \leq Z \leq 3) \approx 0.99.$$

The figure below shows these probabilities as areas under the graph of the standard normal pdf  $\phi(z)$ .

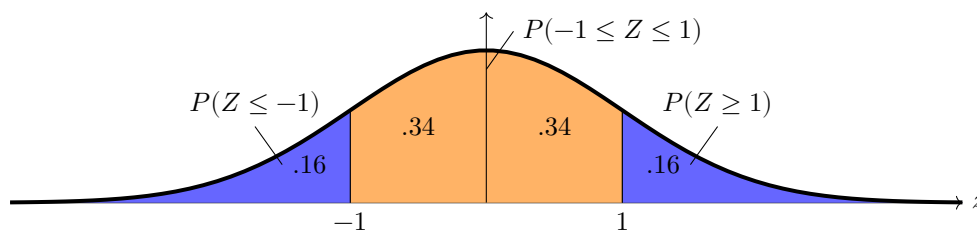


### Symmetry calculations

We can use the symmetry of the standard normal distribution about  $z = 0$  to make some calculations.

**Example 4.** The rule of thumb says  $P(-1 \leq Z \leq 1) \approx 0.68$ . Use this to estimate  $\Phi(1)$ .

**Solution:**  $\Phi(1) = P(Z \leq 1)$ . In the figure, the two tails (in blue) have combined area  $1 - 0.68 = 0.32$ . By symmetry the left tail has area 0.16 (half of 0.32), so  $P(Z \leq 1) \approx 0.68 + 0.16 = 0.84$ .



### 5.2 Using R to compute the standard normal cdf

# Use the R function `pnorm(x,  $\mu$ ,  $\sigma$ )` to compute  $F(x)$  for  $N(\mu, \sigma^2)$

```
pnorm(1,0,1)
[1] 0.8413447
```

```
pnorm(0,0,1)
[1] 0.5
```

```
pnorm(1,0,2)
[1] 0.6914625
```

```
pnorm(1,0,1) - pnorm(-1,0,1)
[1] 0.6826895
```

```
pnorm(5,0,5) - pnorm(-5,0,5)
[1] 0.6826895
```

# Of course  $z$  can be a vector of values

```
pnorm(c(-3,-2,-1,0,1,2,3),0,1)
[1] 0.001349898 0.022750132 0.158655254 0.500000000 0.841344746 0.977249868 0.998650102
```

**Note:** The R function `pnorm`( $x, \mu, \sigma$ ) uses  $\sigma$  whereas our notation for the normal distribution  $N(\mu, \sigma^2)$  uses  $\sigma^2$ .

Here's a table of values with fewer decimal points of accuracy

$z$ :	-2	-1	0	0.3	0.5	1	2	3
$\Phi(z)$ :	0.0228	0.1587	0.5000	0.6179	0.6915	0.8413	0.9772	0.9987

**Example 5.** Use R to compute  $P(-1.5 \leq Z \leq 2)$ .

**Solution:** This is  $\Phi(2) - \Phi(-1.5) = \text{pnorm}(2, 0, 1) - \text{pnorm}(-1.5, 0, 1) = 0.91044$

## 6 Pareto and other distributions

In 18.05, we only have time to work with a few of the many wonderful distributions that are used in probability and statistics. We hope that after this course you will feel comfortable learning about new distributions and their properties when you need them. Wikipedia is often a great starting point.

The Pareto distribution is one common, beautiful distribution that we will not have time to cover in depth.

1. Parameters:  $m > 0$  and  $\alpha > 0$ .
2. Range:  $[m, \infty)$ .
3. Notation: Pareto( $m, \alpha$ ).
4. Density:  $f(x) = \frac{\alpha m^\alpha}{x^{\alpha+1}}$ .
5. Distribution: (easy integral)

$$F(x) = 1 - \frac{m^\alpha}{x^\alpha}, \text{ for } x \geq m$$

6. Tail distribution:  $P(X > x) = m^\alpha/x^\alpha$ , for  $x \geq m$ .
7. Models: The Pareto distribution models a [power law](#), where the probability that an event occurs varies as a power of some attribute of the event. Many phenomena follow a power law, such as the size of meteors, income levels across a population, and population levels across cities. See Wikipedia for loads of examples:

[https://en.wikipedia.org/wiki/Pareto\\_distribution#Applications](https://en.wikipedia.org/wiki/Pareto_distribution#Applications)

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