Class 13 in-class problems, 18.05, Spring 2022

In class examples and discussion

Class example 1.

- *Three types of coins with probabilities* 0.25*,* 0.5*,* 0.75 *of heads.*
- *Assume the numbers of each type are in the ratio 1 to 2 to 1.*
- *Assume we pick a coin at random, toss it twice and get* TT*.*

Compute the posterior probability the coin has probability 0.25 *of heads.*

Solution: Let θ be the probability of heads for the chosen coin. We have three hypotheses: $\theta = 0.25$, $\theta = 0.5$, $\theta = 0.75$. Let's denote these hypotheses by $\theta_{0.25}$, $\theta_{0.5}$ and $\theta_{0.75}$.

The data is TT and we want to compute $P(\theta_{0.25}\,|\,TT)$.

Using a Bayesian update table:

hypotheses $\mathcal{H}$	prior $P(\mathcal{H})$	likelihood $P(TT\|\mathcal{H})$	Bayes numerator $P(TT\|\mathcal{H})P(\mathcal{H})$	posterior $P(\mathcal{H}\|TT)$
$\theta_{0.25}$	1/4	$(0.75)^2$	0.141	0.500
$\theta_{0.5}$	1/2	$(0.5)^2$	0.125	0.444
$\theta_{0.75}$	1/4	$(0.25)^2$	0.016	0.056
Total	1		$P(TT) = 0.281$	1

Don't forget that the update table is just a nice presentation of Bayes' formula and the law of total probability. Please be sure you understand how each of the entries in the update table correspond to the pieces of Bayes' formula. For example, the posterior for $\theta_{0.25}$ is computed in the first row. Written long-hand it looks like:

$$
\begin{aligned}
P(\theta_{0.25}|TT) &= \frac{P(TT|\theta_{0.25})P(\theta_{0.25})}{P(TT)} \\
&= \frac{P(TT|\theta_{0.25})P(\theta_{0.25})}{P(TT|\theta_{0.25})P(\theta_{0.25}) + P(TT|\theta_{0.5})P(\theta_{0.5}) + P(TT|\theta_{0.75})P(\theta_{0.75})} \\
&= \frac{(0.75)^2(1/4)}{(0.75)^2(1/4) + (0.5)^2(1/2) + (0.25)^2(1/4)} \\
&= 0.5.
\end{aligned}
$$

Note. The total probability $P(TT)$ is also called the prior predictive probability of the data.

Concept questions

Concept question 1. Discrete or continuous?

Suppose $X \sim$ *Bernoulli*(θ) *where the value of* θ *is unknown. If we use Bayesian methods to make probabilistic statements about* θ *then which one of the following is true?*

1. *The random variable is discrete, the space of hypotheses is discrete.*

2. *The random variable is discrete, the space of hypotheses is continuous.*

3. *The random variable is continuous, the space of hypotheses is discrete.*

4. *The random variable is continuous, the space of hypotheses is continuous.*

Solution: 2. A Bernoulli random variable takes values 0 or 1. So X is discrete. The parameter θ can be anywhere in the continuous range [0,1]. Therefore the space of hypotheses is continuous.

Board questions

Problem 1. Total probability
(a) *A coin has unknown probability of heads θ with prior pdf, for the value of θ, $f(\theta) = 3\theta^2$. Find the probability of throwing tails on the first toss.*

(b) *Describe an experiment with success and failure that this models. Include the reason for the prior in your description.*

Solution: (a) Take $x = 1$ for heads and $x = 0$ for tails. The likelihood $p(x = 0|\theta) = 1 - \theta$.

The law of total probability says

$$p(x = 0) = \int_0^1 p(x = 0|\theta)f(\theta)\,d\theta = \int_0^1 (1 - \theta)\,3\theta^2\,d\theta = 1/4.$$

(b) There are many possible examples. Here's one:

A medical treatment has unknown probability θ of success. A priori we think it's a good treatment so we use a prior of $f(\theta) = 3\theta^2$ which is biased towards success. If the first use is succesful, then after updating we would believe a little more strongly in the treatment.

Problem 2. Bent coin 1
We have a 'bent' coin with an unknown probability θ of heads. Assume the following:

- *Prior for the value of θ: $f(\theta) = 2\theta$ on $[0, 1]$.*
- *Data: toss once and get heads.*

(a) *Find the posterior pdf to this data.*

(b) *Suppose you toss again and get tails. Update your posterior from part (a) using this data.*

(c) *On one set of axes graph the prior and the posteriors from parts (a) and (b).*

(a) Solution: Here's the update table

hypoth.	range	prior	likelihood	Bayes numerator	posterior
θ	$[0, 1]$	$2\theta\,d\theta$	θ	$2\theta^2\,d\theta$	$3\theta^2\,d\theta$
Total	$[0, 1]$	1		$T = \int_0^1 2\theta^2\,d\theta = 2/3$	1

Posterior pdf: $f(\theta|x) = 3\theta^2$. (Graph below.)

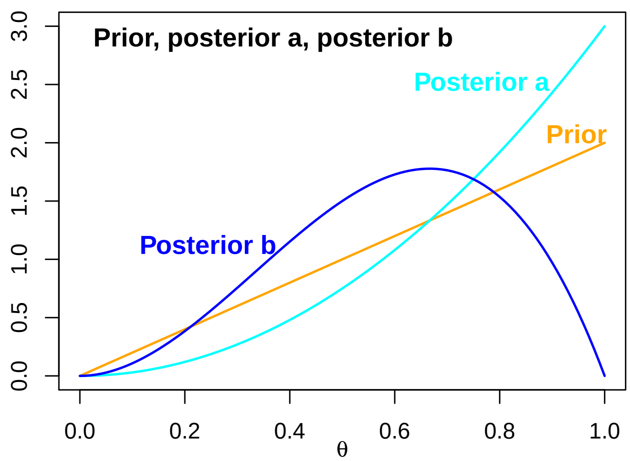
Note: We don't really need to compute T. Once we know the posterior density is of the form $c\theta^2$ we only have to find the value of c makes it have total probability 1.

(b) Solution:

hypoth.	range	prior	likelihood	Bayes numerator	posterior
θ	$[0,1]$	$3\theta^2\,d\theta$	$1-\theta$	$3\theta^2(1-\theta), d\theta$	$12\theta^2(1-\theta)\,d\theta$
Total	$[0,1]$	1		$\int_0^1 3\theta^2(1-\theta)\,d\theta = 1/4$	1

Posterior pdf: $f(\theta|x) = 12\theta^2(1-\theta)$.

(c) Solution: Here is the plot of the prior and the two posteriors. Notice that posterior b is 0 at $\theta = 1$. This happened because the tails on the second toss means θ cannot equal 1.



Problem 2. Bent coin 2

Same scenario: bent coin $\sim$ Bernoulli(θ).

Flat prior: $f(\theta) = 1$ on $[0,1]$

Data: toss 27 times and get 15 heads and 12 tails.

Use this data to find the posterior pdf.

Write an integral formula for the normalizing factor (total probability of the data), but do not compute it. Call its value T and give the posterior pdf in terms of T.

Solution: Here's the update table.

hypoth.	range	prior	likelihood	Bayes numerator	posterior
θ	$[0,1]$	$1\cdot d\theta$	$\binom{27}{15}\theta^{15}(1-\theta)^{12}$	$\binom{27}{15}\theta^{15}(1-\theta)^{12}\,d\theta$	$c\theta^{15}(1-\theta)^{12}\,d\theta$
Total	$[0,1]$	1		$T = \int_0^1 \binom{27}{15}\theta^{15}(1-\theta)^{12}\,d\theta$	1

So, $f(\theta|x) = c\theta^{15}(1-\theta)^{12}$, where $c = \dfrac{\binom{27}{15}}{T}$. A computation (or Wikipedia) would show $c = \dfrac{28!}{15!\,12!}$

Both prior and posterior are Beta distributions. Here are plots.