# Class 23 in-class problems, 18.05, Spring 2022

## Concept questions

### Concept question 1. Overnight polling

*During the presidential election season, pollsters often do 'overnight polls' and report a 'margin of error' of about* $\pm 4\%$.

*The number of people polled is in which of the following ranges?*

*(a) 0 – 50*

*(b) 50 – 100*

*(c) 100 – 500*

*(d) 300 – 600*

*(e) 600 – 1000*

**Solution:** $4\% = 1/25$. So $25 = \sqrt{n} \;\Rightarrow\; n = 625$.

## Board questions

### Problem 1. Confidence intervals for a binomial

*For a poll to find the proportion* $\theta$ *of people supporting X we know that a* $(1-\alpha)$ *confidence interval for* $\theta$ *is given by*

$$\left[\bar{x} - \frac{z_{\alpha/2}}{2\sqrt{n}},\ \bar{x} + \frac{z_{\alpha/2}}{2\sqrt{n}}\right].$$

***(a)*** *How many people would you have to poll to have a margin of error of* 0.01 *with 95% confidence? (You can do this in your head.)*

***(b)*** *How many people would you have to poll to have a margin of error of* 0.01 *with 80% confidence. (You'll want R or other calculator here.)*

***(c)*** *If* $n = 900$, *compute the 95% and 80% confidence intervals for* $\theta$.

**Solution:** **(a)** Need $1/\sqrt{n} = 0.01$ So $n = 10000$.

**(b)** $\alpha = 0.2$, so $z_{\alpha/2} =$ `qnorm(0.9)` $= 1.2816$. So we need $\dfrac{z_{\alpha/2}}{2\sqrt{n}} = 0.01$. This gives $n = 4106$.

**(c)** Rule-of-thumb 95% interval: $\overline{x} \pm \dfrac{1}{\sqrt{n}} = \overline{x} \pm \dfrac{1}{30} = \overline{x} \pm 0.0333$.

80% interval: $\overline{x} \pm z_{0.1}\dfrac{1}{2\sqrt{n}} = \overline{x} \pm 1.2816 \cdot \dfrac{1}{60} = \overline{x} \pm 0.021$.

### Problem 2. Pivoting: confidence intervals and non-rejection regions

*This question gets at the relationship between confidence intervals and non-rejection regions.*

**Main point:** *For a sample with sample mean* $\overline{x}$, *the confidence interval consists of all values* $\mu$ *for which a NHST with null hypothesis mean* $= \mu$ *would not reject on seeing* $\overline{x}$.

*Assume we have independent data* $x_1, \ldots, x_n \sim N(\mu, \sigma^2)$, *where* $\mu$ *is unknown and* $\sigma$ *is known.*

***(a)*** *For null hypothesis $\mu = \mu_0$ give the two-sided non-rejection region for significance level $\alpha$.*

***(b)*** *Call the data average $\overline{x}$. Give the $1-\alpha$ confidence interval for $\mu$.*

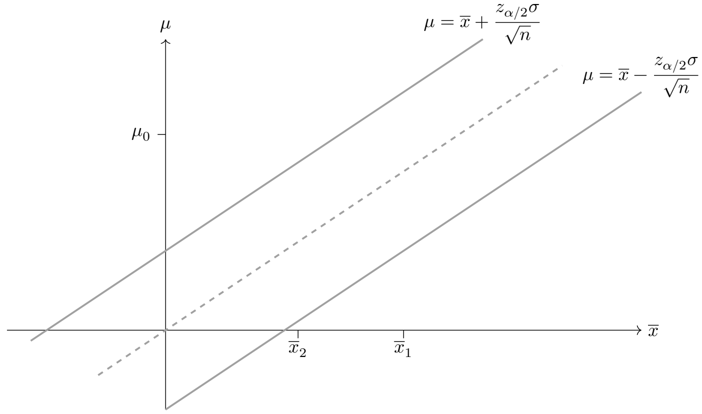
***(c)*** *Use the $\overline{x}, \mu$-plane below. Note the conveniently included guides.*

*(i) Plot the horizontal line segment at height $\mu_0$ showing the non-rejection region for $H_0 : \mu = \mu_0$ (significance level = $\alpha$).*

*(ii) Plot the horizontal line segment at other heights showing the non-rejection region for the corresponding $\mu$.*

*(iii) Plot the vertical line segments showing the $1-\alpha$ confidence intervals around $\overline{x}_1$ and $\overline{x}_2$*

*(iv) Plot the vertical line segment at other values of $\overline{x}$ showing the corresponding confidence interval.*

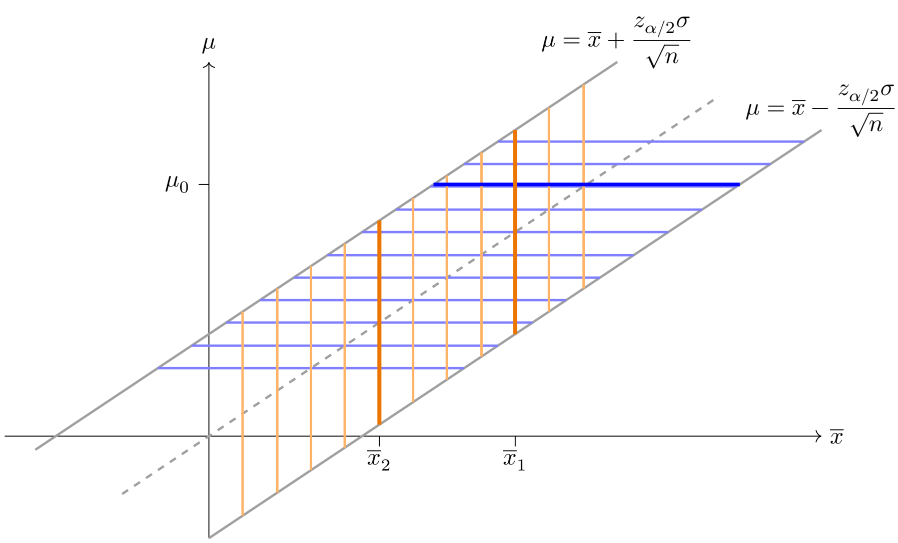



*Understand how the main point connects with your graph.*

**Solution:** **(a)** $[\mu_0 - \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}, \mu_0 + \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}]$.

**(b)** $[\overline{x} - \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}, \overline{x} + \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}]$.

**(c)**



Both horizontal and vertical segments run between the same guides. So, every $\mu$ in the confidence interval for $\overline{x}$ has $\overline{x}$ in its non-rejection region. That is, the confidence interval consists of all $\mu$ that would not reject on seeing data $\overline{x}$.

Said differently, if the horizontal non-rejection region based on $\mu$ intersects the vertical confidence interaval based on $\overline{x}$. Then both of the following statements are true.

1. $\overline{x}$ is in the non-rejection region (based on $\mu$).
2. $\mu$ is in the confidence interval (based on $\overline{x}$).

For example, $\overline{x}_1$ is in the non-rejection region for $\mu_0$ and $\mu_0$ is in the confidence interval for $\overline{x}_1$. Likewise $\mu_0$ and $\overline{x}_2$ are not in each other's intervals.

**Problem 3. Exact binomial confidence interval**
*This was not used in class, but it is a nice problem, so we included it here.*
*Use this table of binomial(8,θ) probabilities to:*

**1.** *find the (two-sided) rejection region with significance level* $0.10$ *for each value of* $\theta$.

**2.** *Given* $x = 7$, *find the 90% confidence interval for* $\theta$.

**3.** *Repeat for* $x = 4$.

| $\theta\backslash x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.430 | 0.383 | 0.149 | 0.033 | 0.005 | 0.000 | 0.000 | 0.000 | 0.000 |
| 0.3 | 0.058 | 0.198 | 0.296 | 0.254 | 0.136 | 0.047 | 0.010 | 0.001 | 0.000 |
| 0.5 | 0.004 | 0.031 | 0.109 | 0.219 | 0.273 | 0.219 | 0.109 | 0.031 | 0.004 |
| 0.7 | 0.000 | 0.001 | 0.010 | 0.047 | 0.136 | 0.254 | 0.296 | 0.198 | 0.058 |
| 0.9 | 0.000 | 0.000 | 0.000 | 0.000 | 0.005 | 0.033 | 0.149 | 0.383 | 0.430 |

**Solution:** For each $\theta$, the non-rejection region is blue, the rejection region is orange. In each row, the rejection region has probability at most $\alpha = 0.10$.

| $\theta\backslash x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.430 | 0.383 | 0.149 | 0.033 | 0.005 | 0.000 | 0.000 | 0.000 | 0.000 |
| 0.3 | 0.058 | 0.198 | 0.296 | 0.254 | 0.136 | 0.047 | 0.010 | 0.001 | 0.000 |
| 0.5 | 0.004 | 0.031 | 0.109 | 0.219 | 0.273 | 0.219 | 0.109 | 0.031 | 0.004 |
| 0.7 | 0.000 | 0.001 | 0.010 | 0.047 | 0.136 | 0.254 | 0.296 | 0.198 | 0.058 |
| 0.9 | 0.000 | 0.000 | 0.000 | 0.000 | 0.005 | 0.033 | 0.149 | 0.383 | 0.430 |

For $x = 7$ the 90% confidence interval for $\theta$ is $[0.7, 0.9]$. These are the values of $\theta$ we wouldn't reject as null hypotheses. They are the blue entries in the $x = 7$ column.

For $x = 4$ the 90% confidence interval for $\theta$ is $[0.3, 0.7]$.

**Problem 4. Pivoting: Chi square confidence intervals for variance**
*This was not used in class, but it is a nice problem, so we included it here.*
*Assume we have independent data* $x_1, \dots, x_n \sim N(\mu, \sigma^2_{true})$, *where* $\sigma_{true}$ *is unknown and our parameter of interest.*

*Let* $s^2$ *be the sample variance. We know that* $\dfrac{(n-1)s^2}{\sigma^2_{true}} \sim \chi^2(n-1)$. *Thus,*

$$P(c_{1-\alpha/2} \le \frac{(n-1)s^2}{\sigma^2} \le c_{\alpha/2}\,|, \sigma_{true} = \sigma) = 1 - \alpha.$$

*Here, $c_{\alpha/2}$ is the right critical point for the $\chi^2(n-1)$ distribution.*

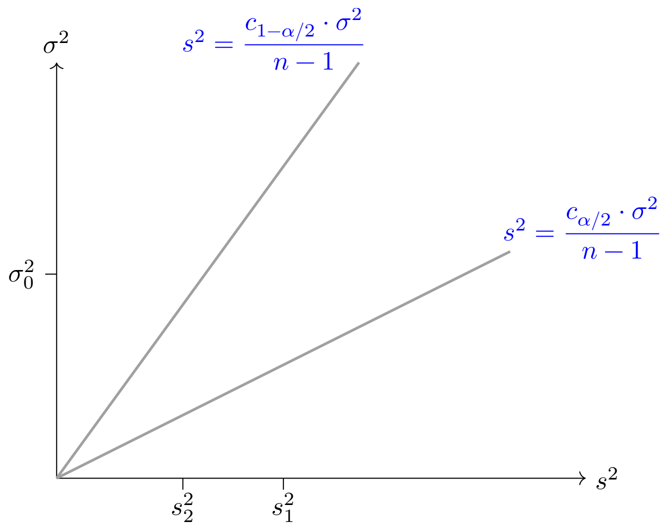
*Using this, for a two-sided significance test with with $H_0 : \sigma_{true} = \sigma$, the non-rejection region for $s^2$ at significance level $\alpha$ is*

$$\frac{c_{1-\alpha/2}\,\sigma^2}{n-1} \le s^2 \le \frac{c_{\alpha/2}\,\sigma^2}{n-1}$$

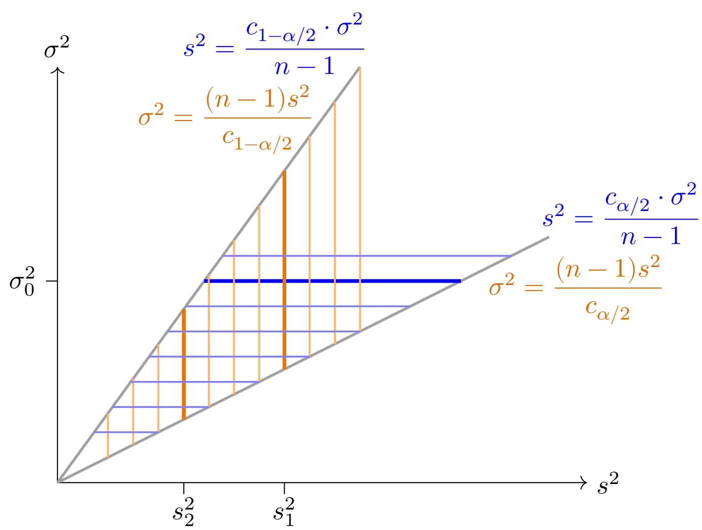
*Pivoting, we get the $1-\alpha$ confidence interval for $\sigma^2$ produced by the data is*

$$\frac{(n-1)s^2}{c_{\alpha/2}} \le \sigma^2 \le \frac{(n-1)s^2}{c_{1-\alpha/2}}$$

*Display this graphically on the $\sigma^2$-$s^2$ axes shown.*




**Solution:** In the graph below, the blue horizontal segements at height $\sigma^2$ is the non-rejection region for $H_0 : \sigma^2_{\text{true}} = \sigma^2$, i.e. it is a range of $s^2$. The vertical orange segment over $s^2$ is the confidence interval produced by data $s^2$.




The horizontal and vertical segments run between the same guides –this is the geometric meaning of pivoting. Pivoting is seen algebraically by changing the formulas for the guides from $s^2$ as a function of $\sigma^2$ to $\sigma^2$ as a function of $s^2$.

So, the confidence interval produced by data $s^2$ consists of all $\sigma^2$ for which the null hypothesis $\sigma^2_{\text{true}} = \sigma^2$ will not reject (upon seeing the data $s^2$).

Said differently, if the horizontal non-rejection region based on $H_0 : \sigma^2_{\text{true}} = \sigma^2$ intersects the vertical confidence interval based on $s^2$. Then both of the following statements are true.

1. $s^2$ is in the non-rejection region (based on $H_0 : \sigma^2_{\text{true}} = \sigma^2$).
2. $\sigma^2$ is in the confidence interval (based on data $s^2$).

For example, $s_1^2$ is in the non-rejection region for $H_0 : \sigma^2_{\text{true}} = \sigma_0^2$ and $\sigma_0^2$ is in the confidence interval produced by data $s_1^2$. Likewise $\sigma_0^2$ and $s_2^2$ are not in each other's intervals.