Joint Distributions, Independence, Covariance and Correlation
18.05 Spring 2022

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Announcements/Agenda

Announcements

- Exam 1 on Thursday March 10. Covers classes 1-7.
- Designed for 1 hour. You will have the full 80 minutes.
- Review materials will be posted tomorrow.
- Class on Tuesday 3/8 will be mostly review.
- For the exam you will be given a table of standard normal probabilities.
- You can bring in a cheat sheet: 1 side of an 8 × 11 sheet of paper. You’ll turn it in with the exam for 5 points.

Agenda

- Joint distributions: pmf, pdf, cdf
- Marginal distributions
- Independence
- Covariance and correlation
Joint Distributions

$X$ and $Y$ are jointly distributed random variables.

Discrete: Probability mass function (pmf): $p(x_i, y_j)$

Continuous: probability density function (pdf): $f(x, y)$

Both: cumulative distribution function (cdf):
$F(x, y) = P(X \leq x, Y \leq y)$
Discrete joint pmf: example 1

Roll two dice: \( X = \# \) on first die, \( Y = \# \) on second die

\( X \) takes values in 1, 2, ..., 6, \( Y \) takes values in 1, 2, ..., 6

Joint probability table:

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pmf: \( p(i, j) = 1/36 \) for any \( i \) and \( j \) between 1 and 6.
Discrete joint pmf: example 2

Roll two dice: \( X = \# \) on first die, \( T = \) sum of both dice

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E.g. \( p(4, 2) = 0, \ p(1, 7) = 1/36 \).
Continuous joint distributions

- $X$ takes values in $[a, b]$, $Y$ takes values in $[c, d]$
- $(X, Y)$ takes values in $[a, b] \times [c, d]$.
- Joint probability density function (pdf) $f(x, y)$

$$f(x, y) \, dx \, dy = \text{probability of being in the small square around } (x, y).$$
Properties of the joint pmf and pdf

Discrete case: probability mass function (pmf)
1. \(0 \leq p(x_i, y_j) \leq 1\)
2. Total probability is 1.

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1
\]

Continuous case: probability density function (pdf)
1. \(0 \leq f(x, y)\)
2. Total probability is 1.

\[
\int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy = 1
\]

Note: \(f(x, y)\) can be greater than 1: it is a density not a probability.
• You should understand double integrals conceptually as double sums.

• You should be able to compute double integrals over rectangles.

• For a non-rectangular region, when \( f(x, y) = c \) is constant, you should know that the double integral is the same as \( c \times (\text{the area of the region}) \).

• You should be able to compute partial derivatives.
Example: discrete events
Roll two dice: $X = \#$ on first die, $Y = \#$ on second die.
Consider the event: $A = 'Y - X \geq 2'$
Describe the event $A$ and find its probability.
Example: discrete events

Roll two dice: \( X = \# \) on first die, \( Y = \# \) on second die.

Consider the event: \( A = \{Y - X \geq 2\} \)

Describe the event \( A \) and find its probability.

**Solution:** We can describe \( A \) as a set of \((X, Y)\) pairs:

\[
A = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 5), (3, 6), (4, 6)\}.
\]

Or we can visualize it by shading the table:

\[
\begin{array}{c|ccccccc}
X \backslash Y & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
2 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
3 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
4 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
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6 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 & 1/36 \\
\end{array}
\]

\[
P(A) = \text{sum of probabilities in shaded cells} = \frac{10}{36}.
\]
Example: continuous events

Suppose \((X, Y)\) takes values in \([0, 1] \times [0, 1]\).

Uniform density \(f(x, y) = 1\).

Visualize the event ‘\(X > Y\)’ and find its probability.
Example: continuous events

Suppose \((X, Y)\) takes values in \([0, 1] \times [0, 1]\).
Uniform density \(f(x, y) = 1\).

Visualize the event ‘\(X > Y\)’ and find its probability.

**Solution:**

The event takes up half the square. Since the density is uniform this is half the probability. That is, \(P(X > Y) = 0.5\).
Cumulative distribution function

\[ F(x, y) = P(X \leq x, Y \leq y) = \int_{c}^{y} \int_{a}^{x} f(u, v) \, du \, dv. \]

(a and c are the bottom of the ranges of X and Y respectively.)

\[ f(x, y) = \frac{\partial^2 F}{\partial x \partial y}(x, y). \]

Properties

1. \( F(x, y) \) is non-decreasing. That is, as \( x \) or \( y \) increases \( F(x, y) \) increases or remains constant.

2. \( F(x, y) = 0 \) at the lower left of its range.

3. \( F(x, y) = 1 \) at the upper right of its range.
## Marginal pmf and pdf

Roll two dice: $X = \#$ on first die, $T =$ total on both dice.

The pmf of $X$ is found by summing the rows. The pmf of $T$ is found by summing the columns. These are called marginal pmfs of the joint distribution.

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For continuous distributions the marginal pdf $f_X(x)$ is found by integrating out the $y$. Likewise for $f_Y(y)$. 
Suppose $X$ and $Y$ are random variables and
- $(X, Y)$ takes values in $[0, 1] \times [0, 1]$.
- the pdf is $f(x, y) = x + y$.

(a) Show $f(x, y)$ is a valid pdf.

(b) Visualize the event $A = \{X > 0.3 \text{ and } Y > 0.5\}$. Find its probability.

(c) Find the cdf $F(x, y)$.

(d) Use the cdf $F(x, y)$ to find the marginal cdf $F_X(x)$ and $P(X < 0.5)$.

(e) Find the marginal pdf $f_X(x)$. Use this to find $P(X < 0.5)$.

(f) See next slide
(f) (New scenario) From the following table compute $F(3.5, 4)$.

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Independence

Events $A$ and $B$ are independent if

$$P(A \cap B) = P(A)P(B).$$

Random variables $X$ and $Y$ are independent if

$$F(x, y) = F_X(x)F_Y(y).$$

Discrete random variables $X$ and $Y$ are independent if

$$p(x_i, y_j) = p_X(x_i)p_Y(y_j).$$

Continuous random variables $X$ and $Y$ are independent if

$$f(x, y) = f_X(x)f_Y(y).$$

Independence means probabilities multiply!
Concept question: Independence I

Roll two dice: \( X = \) value on first, \( Y = \) value on second

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Are \( X \) and \( Y \) independent? 1. Yes 2. No
**Concept question: Independence II**

Roll two dice: \( X = \text{value on first}, \quad T = \text{sum} \)

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Are \(X\) and \(Y\) independent? 1. Yes 2. No
Concept question: Independence III

Which of the following joint pdfs are the variables independent? (Each of the ranges is a rectangle chosen so that $\int \int f(x, y) \, dx \, dy = 1$.)

(i) $f(x, y) = 4x^2y^3$.

(ii) $f(x, y) = \frac{1}{2}(x^3y + xy^3)$.

(iii) $f(x, y) = 6e^{-3x-2y}$

(a) i  (b) ii  (c) iii  (d) i, ii

(e) i, iii  (f) ii, iii  (g) i, ii, iii  (h) None
Covariance

Measures the degree to which two random variables vary together, e.g. height and weight of people.

$X, Y$ random variables with means $\mu_X$ and $\mu_Y$.

**Definition:** $\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$.

- **Positive covariance:** When $X$ is bigger than $\mu_X$ then $Y$ is ‘usually’ bigger than $\mu_Y$, and vice versa
- **Negative covariance:** When $X$ is bigger than $\mu_X$ then $Y$ is ‘usually’ smaller than $\mu_Y$, and vice versa
- **Zero covariance:** The sign of $X - \mu_X$ tells us nothing about the sign of $(Y - \mu_Y)$. 
Properties of covariance

1. $\text{Cov}(aX + b, cY + d) = ac \text{Cov}(X, Y)$ for constants $a, b, c, d$.

2. $\text{Cov}(X_1 + X_2, Y) = \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y)$.

3. $\text{Cov}(X, X) = \text{Var}(X)$


5. If $X$ and $Y$ are independent then $\text{Cov}(X, Y) = 0$.

6. **Warning**, the converse is not true: if covariance is 0 the variables might not be independent.
Table question

Suppose we have the following joint probability table.

<table>
<thead>
<tr>
<th>Y \ X</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>p(y_j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>1</td>
<td>1/4</td>
<td>0</td>
<td>1/4</td>
<td>1/2</td>
</tr>
<tr>
<td>p(x_i)</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
<td>1</td>
</tr>
</tbody>
</table>

At your table work out the covariance Cov(X, Y).

Are X and Y independent?
Table question

Suppose we have the following joint probability table.

\[
\begin{array}{ccc|c}
Y \backslash X & -1 & 0 & 1 \\
0 & 0 & 1/2 & 0 & 1/2 \\
1 & 1/4 & 0 & 1/4 & 1/2 \\
\end{array}
\]

\[
\begin{array}{ccc|c}
p(x_i) & 1/4 & 1/2 & 1/4 & 1 \\
\end{array}
\]

At your table work out the covariance \( \text{Cov}(X, Y) \).

Are \( X \) and \( Y \) independent?

\textbf{Covariance} = 0. \textbf{Not independent}!

\textbf{Key point}: covariance measures the linear relationship between \( X \) and \( Y \). It can completely miss a quadratic or higher order relationship.
Correlation

Like covariance, but removes scale.
The correlation coefficient between $X$ and $Y$ is defined by

$$
\text{Cor}(X, Y) = \rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.
$$

Properties:

1. $\rho$ is the covariance of the standardized versions of $X$ and $Y$.
2. $\rho$ is dimensionless (it’s a ratio).
3. $-1 \leq \rho \leq 1$.
4. $\rho = 1$ if and only if $Y = aX + b$ with $a > 0$
5. $\rho = -1$ if and only if $Y = aX + b$ with $a < 0$. 
Flip a fair coin 11 times. (The tosses are all independent.)

Let $X =$ number of heads in the first 6 flips

Let $Y =$ number of heads on the last 6 flips.

Compute $\text{Cov}(X, Y)$ and $\text{Cor}(X, Y)$. 
Real-life correlations

- Over time, amount of ice cream consumption is correlated with number of pool drownings.

- In 1685 (and today) being a student is the most dangerous profession. That is, the average age of those who die is less than any other profession.

- In 90% of bar fights ending in a death the person who started the fight died.

- Hormone replacement therapy (HRT) is correlated with a lower rate of coronary heart disease (CHD).
Correlation is not causation

Edward Tufte: "Empirically observed covariation is a necessary but not sufficient condition for causality."
Overlapping sums of uniform random variables

We made two random variables $X$ and $Y$ from overlapping sums of uniform random variables.

For example:

\[
X = X_1 + X_2 + X_3 + X_4 + X_5 \\
Y = X_3 + X_4 + X_5 + X_6 + X_7
\]

These are sums of 5 of the $X_i$ with 3 in common.

If we sum $r$ of the $X_i$ with $s$ in common we name it $(r, s)$.

Below are a series of scatterplots produced using R.
Scatter plots

(1, 0) cor=0.00, sample_cor=-0.07

(5, 1) cor=0.20, sample_cor=0.21

(2, 1) cor=0.50, sample_cor=0.48

(10, 8) cor=0.80, sample_cor=0.81
Intuition check

Toss a fair coin $2n + 1$ times. Let $X$ be the number of heads on the first $n + 1$ tosses and $Y$ the number on the last $n + 1$ tosses.

If $n = 1000$ then $\text{Cov}(X, Y)$ is:

(a) 0  (b) 1/4  (c) 1/2  (d) 1

(e) More than 1  (f) tiny but not 0

(This is computed in the answer to the next board question.)
Toss a fair coin $2n + 1$ times. Let $X$ be the number of heads on the first $n + 1$ tosses and $Y$ the number on the last $n + 1$ tosses.

Compute $\text{Cov}(X, Y)$ and $\text{Cor}(X, Y)$. 