Bayesian Updating: Discrete Priors: 18.05 Spring 2022

\[
P(\text{I'm near the ocean} \mid \text{I picked up a seashell}) = \frac{P(\text{I picked up a seashell} \mid \text{I'm near the ocean}) P(\text{I'm near the ocean})}{P(\text{I picked up a seashell})}
\]

Statistically speaking, if you pick up a seashell and don't hold it to your ear, you can probably hear the ocean.

Image courtesy of xkcd. License: CC BY-NC.

https://xkcd.com/1236/
Announcements

- Pset 6 is due tomorrow. Pset 7 will be due on Monday April 4.
- R Studio 4 is graded. Good job overall.
  - A few people used the sample mean and variance to draw the normal curve – the instructions asked for the theoretical values.
  - Coding hint: Don’t keep retyping values like 18/38 and 20/38. Put them in a variable and use that.

Agenda

- Bayesian updating from prior to posterior probabilities.
- Organizing the computation in tables.
- Today: Known priors (base rates) – not like in real science.
Concept question: Learning from experience

(a) Which treatment would you choose?

1. Treatment 1: cured 100% of patients in a trial.
2. Treatment 2: cured 95% of patients in a trial.
3. Treatment 3: cured 90% of patients in a trial.
Concept question: Learning from experience

(a) Which treatment would you choose?
1. Treatment 1: cured 100% of patients in a trial.
2. Treatment 2: cured 95% of patients in a trial.
3. Treatment 3: cured 90% of patients in a trial.

(b) Which treatment would you choose?
1. Treatment 1: cured 3 out of 3 patients in a trial.
2. Treatment 2: cured 19 out of 20 patients treated in a trial.
Which die is it?

- I have a bag containing dice of two types: 4-sided and 6-sided.
- Suppose I pick a die at random.
- What are the hypotheses for which die I chose?
- I’ll roll the die.
- Based on what I rolled which type would you guess I picked?
Which die is it?

- I have a bag containing dice of two types: 4-sided and 6-sided.
- Suppose I pick a die at random.
- What are the hypotheses for which die I chose?
- I’ll roll the die.
- Based on what I rolled which type would you guess I picked?

- Suppose you find out that the bag contained one 4-sided die and 999 6-sided dice. Does this change your guess?
Bayesian updating terminology, trees, tables

**Example.** Bag with one 4-sided die and 999 6-sided dice. Pick one at random and roll it. Suppose I get a 3.

For this example, all of the following are given in the posted solutions to today’s problems.

Terminology: data, hypotheses, likelihoods, prior probabilities, posterior probabilities.

For posterior probability:

\[ P(\mathcal{H}_4 | R = 3) = \frac{P(R = 3 | \mathcal{H}_4) \cdot P(\mathcal{H}_4)}{P(R = 3)} \]

Total probability of the data

Updating using trees and tables.
Board Question: Updating from data

- A certain disease has a prevalence of 0.005.
- A screening test has 2% false positives and 1% false negatives.

Suppose a random patient is screened and has a positive test.

(a) Represent this information with a tree and use Bayes’ theorem to compute the probabilities the patient does and doesn’t have the disease.

(b) Identify the data, hypotheses, likelihoods, prior probabilities and posterior probabilities.

(c) Make a full likelihood table containing all hypotheses and possible test data.

(d) Redo the computation using a Bayesian update table. Match the terms in your table to the terms in your previous calculation.
Board Question: Dice

Five dice: 4-sided, 6-sided, 8-sided, 12-sided, 20-sided.

I pick one at random, roll it and report that the roll was a 13.

Goal: Find the probabilities the die is 4, 6, 8, 12 or 20 sided.

(a) Identify the hypotheses.

(b) Make a likelihood table with columns for the data ‘rolled a 13’, ‘rolled a 5’ and ‘rolled a 9’.

(c) Make a Bayesian update table and compute the posterior probabilities that the chosen die is each of the five dice.

(d) Same question if I had reported a 5.

(e) Same question if I had reported a 9.

(Keep the tables for 5 and 9 handy for the next problem!)
Tabular solution

\( \mathcal{D} = \text{‘rolled a 13’} \)

<table>
<thead>
<tr>
<th>hypothesis</th>
<th>prior</th>
<th>likelihood</th>
<th>Bayes numerator</th>
<th>posterior</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{H}_4 )</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{H}_6 )</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{H}_8 )</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{H}_{12} )</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{H}_{20} )</td>
<td>1/5</td>
<td>1/20</td>
<td>1/100</td>
<td>1</td>
</tr>
<tr>
<td>total</td>
<td>1</td>
<td>1/100</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
**Tabular solution**

\[ \mathcal{D} = \text{`rolled a 5’} \]

<table>
<thead>
<tr>
<th>hypothesis</th>
<th>prior</th>
<th>likelihood</th>
<th>Bayes numerator</th>
<th>posterior</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{H}_4 )</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{H}_6 )</td>
<td>1/5</td>
<td>1/6</td>
<td>1/30</td>
<td>0.392</td>
</tr>
<tr>
<td>( \mathcal{H}_8 )</td>
<td>1/5</td>
<td>1/8</td>
<td>1/40</td>
<td>0.294</td>
</tr>
<tr>
<td>( \mathcal{H}_{12} )</td>
<td>1/5</td>
<td>1/12</td>
<td>1/60</td>
<td>0.196</td>
</tr>
<tr>
<td>( \mathcal{H}_{20} )</td>
<td>1/5</td>
<td>1/20</td>
<td>1/100</td>
<td>0.118</td>
</tr>
<tr>
<td>total</td>
<td>1</td>
<td>0.085</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>


**Tabular solution**

\( \mathcal{D} = \text{‘rolled a 9’} \)

<table>
<thead>
<tr>
<th>hypothesis</th>
<th>prior</th>
<th>likelihood</th>
<th>Bayes numerator</th>
<th>posterior</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{H} )</td>
<td>( P(\mathcal{H}) )</td>
<td>( P(\mathcal{D}</td>
<td>\mathcal{H}) )</td>
<td>( P(\mathcal{D}</td>
</tr>
<tr>
<td>( \mathcal{H}_4 )</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{H}_6 )</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{H}_8 )</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{H}_{12} )</td>
<td>1/5</td>
<td>1/12</td>
<td>1/60</td>
<td>0.625</td>
</tr>
<tr>
<td>( \mathcal{H}_{20} )</td>
<td>1/5</td>
<td>1/20</td>
<td>1/100</td>
<td>0.375</td>
</tr>
<tr>
<td>total</td>
<td>1</td>
<td>0.0267</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

"Tabular solution"
Iterated Updates

Suppose I rolled a 5 and then a 9.

Update in two steps:

First for the 5

Then update the update for the 9.
Tabular solution

\(D_1 = \text{‘rolled a 5’}\)
\(D_2 = \text{‘rolled a 9’}\)
Bayes numerator_1 = likelihood_1 \times \text{prior}.
Bayes numerator_2 = likelihood_2 \times \text{Bayes numerator}_1

<table>
<thead>
<tr>
<th>hyp.</th>
<th>prior</th>
<th>likel. 1</th>
<th>Bayes</th>
<th>likel. 2</th>
<th>Bayes</th>
<th>posterior</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H)</td>
<td>(P(H))</td>
<td>(P(D_1</td>
<td>H))</td>
<td>(<em>,</em>,*)</td>
<td>(P(D_2</td>
<td>H))</td>
</tr>
<tr>
<td>(H_4)</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(H_6)</td>
<td>1/5</td>
<td>1/6</td>
<td>1/30</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(H_8)</td>
<td>1/5</td>
<td>1/8</td>
<td>1/40</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(H_{12})</td>
<td>1/5</td>
<td>1/12</td>
<td>1/60</td>
<td>1/12</td>
<td>1/720</td>
<td>0.735</td>
</tr>
<tr>
<td>(H_{20})</td>
<td>1/5</td>
<td>1/20</td>
<td>1/100</td>
<td>1/20</td>
<td>1/2000</td>
<td>0.265</td>
</tr>
<tr>
<td>total</td>
<td>1</td>
<td></td>
<td>0.0019</td>
<td></td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
Board Question: Iterated updates

Suppose I rolled a 9 and then a 5.

(a) Do the Bayesian update in two steps:
Step 1: First update for the 9.
Step 2: Then update the update for the 5.

(b) Do the Bayesian update in one step.
That is, the data is $\mathcal{D} = '9$ followed by $5'$
Tabular solution: two steps

\( \mathcal{D}_1 = \text{‘rolled a 9’} \)
\( \mathcal{D}_2 = \text{‘rolled a 5’} \)

Bayes numerator_1 = likelihood_1 \times \text{prior.}
Bayes numerator_2 = likelihood_2 \times \text{Bayes numerator}_1

\[
\begin{array}{cccccc}
\text{hyp.} & \text{prior} & \text{likel. 1} & \text{Bayes num. 1} & \text{likel. 2} & \text{Bayes num. 2} & \text{posterior} \\
\mathcal{H} & P(\mathcal{H}) & P(\mathcal{D}_1 | \mathcal{H}) & * * * & P(\mathcal{D}_2 | \mathcal{H}) & * * * & P(\mathcal{H} | \mathcal{D}_1, \mathcal{D}_2) \\
\mathcal{H}_4 & 1/5 & 0 & 0 & 0 & 0 & 0 \\
\mathcal{H}_6 & 1/5 & 0 & 0 & 1/6 & 0 & 0 \\
\mathcal{H}_8 & 1/5 & 0 & 0 & 1/8 & 0 & 0 \\
\mathcal{H}_{12} & 1/5 & 1/12 & 1/60 & 1/12 & 1/720 & 0.735 \\
\mathcal{H}_{20} & 1/5 & 1/20 & 1/100 & 1/20 & 1/2000 & 0.265 \\
\text{total} & 1 & 0.0019 & 1 \\
\end{array}
\]
Tabular solution: one step

\( \mathcal{D} = \text{‘rolled a 9 then a 5’} \)

<table>
<thead>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>( \mathcal{H}_8 )</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \mathcal{H}_{12} )</td>
<td>1/5</td>
<td>1/144</td>
<td>1/720</td>
<td>0.735</td>
</tr>
<tr>
<td>( \mathcal{H}_{20} )</td>
<td>1/5</td>
<td>1/400</td>
<td>1/2000</td>
<td>0.265</td>
</tr>
<tr>
<td>total</td>
<td>1</td>
<td>0.0019</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Along with finding posterior probabilities of hypotheses. We might want to make posterior predictions about the next roll.

With the same setup as before let:
\( \mathcal{D}_1 = \) result of first roll, \( \mathcal{D}_2 = \) result of second roll

(a) Find \( P(\mathcal{D}_1 = 5) \).

(b) Find \( P(\mathcal{D}_2 = 4|\mathcal{D}_1 = 5) \).
Solution

\[ \mathcal{D}_1 = \text{‘rolled a 5’}, \quad \mathcal{D}_2 = \text{‘rolled a 4’} \]

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\text{hyp.} & \text{prior} & \text{likel. 1} & \text{num. 1} & \text{post. 1} & \text{likel. 2} & \text{post. 1 \times likel. 2} \\
\hline
\mathcal{H}_4 & 1/5 & 0 & 0 & 0 & * & 0 \\
\mathcal{H}_6 & 1/5 & 1/6 & 1/30 & 0.392 & 1/6 & 0.392 \cdot 1/6 \\
\mathcal{H}_8 & 1/5 & 1/8 & 1/40 & 0.294 & 1/8 & 0.294 \cdot 1/40 \\
\mathcal{H}_{12} & 1/5 & 1/12 & 1/60 & 0.196 & 1/12 & 0.196 \cdot 1/12 \\
\mathcal{H}_{20} & 1/5 & 1/20 & 1/100 & 0.118 & 1/20 & 0.118 \cdot 1/20 \\
\hline
\text{total} & 1 & 0.085 & 1 & & & 0.124 \\
\hline
\end{array}
\]

The law of total probability tells us \( P(\mathcal{D}_1) \) is the sum of the Bayes numerator 1 column in the table: \( P(\mathcal{D}_1) = 0.085 \).

The law of total probability tells us \( P(\mathcal{D}_2|\mathcal{D}_1) \) is the sum of the last column in the table: \( P(\mathcal{D}_2|\mathcal{D}_1) = 0.124 \).