Confidence Intervals for Normal Data
18.05 Spring 2022

\[
\begin{align*}
\mu_0 \pm 1 & \quad \text{interval centered on } \mu_0 \text{ does not contain } \bar{x} \\
\bar{x} \pm 1 & \quad \text{interval centered on } \bar{x} \text{ does not contain } \mu_0 \\
\mu_0 \pm 2.3 & \quad \text{interval centered on } \mu_0 \text{ contains } \bar{x} \\
\bar{x} \pm 2.3 & \quad \text{interval centered on } \bar{x} \text{ contains } \mu_0
\end{align*}
\]
Exam 2

- Next Thursday in class
- Covers classes 10-20
- You can use a cheat sheet - 1 side of a 8 × 11 sheet
- We’ll give you probability tables – these are in the review materials
- Class on Tuesday will be review
- Review materials are on MITx

Agenda

- Review of critical values and quantiles.
- Computing $z$, $t$, $\chi^2$ confidence intervals for normal data.
- Conceptual view of confidence intervals.
- Confidence intervals for polling (Bernoulli distributions).
Review of critical values and quantiles

- **Quantile:** left tail $P(X < q_\alpha) = \alpha$
- **Critical value:** right tail $P(X > c_\alpha) = \alpha$

Letters for critical values:
- $z_\alpha$ for $N(0, 1)$
- $t_\alpha$ for $t$ distributions
- $c_\alpha$, $x_\alpha$ all purpose

$q_\alpha$ and $z_\alpha$ for the standard normal distribution.
Concept question: critical values

1. \( z_{0.025} = \)
   
   (a) -1.96  (b) -0.95  (c) 0.95  (d) 1.96  (e) 2.87
Concept question: critical values

1. $z_{0.025} =$
   (a) -1.96    (b) -0.95    (c) 0.95    (d) 1.96    (e) 2.87

2. $-z_{0.16} =$
   (a) -1.33    (b) -0.99    (c) 0.99    (d) 1.33    (e) 3.52
Computing confidence intervals from normal data

Suppose the data $x_1, \ldots, x_n$ is independently drawn from $N(\mu, \sigma^2)$

Confidence level $= 1 - \alpha$

- $z$ confidence interval for the mean ($\sigma$ known)

$$\left[ \bar{x} - \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}}, \quad \bar{x} + \frac{z_{\alpha/2} \cdot \sigma}{\sqrt{n}} \right]$$

- $t$ confidence interval for the mean ($\sigma$ unknown)

$$\left[ \bar{x} - \frac{t_{\alpha/2} \cdot s}{\sqrt{n}}, \quad \bar{x} + \frac{t_{\alpha/2} \cdot s}{\sqrt{n}} \right]$$

- $\chi^2$ confidence interval for $\sigma^2$

$$\left[ \frac{n-1}{c_{\alpha/2}} s^2, \quad \frac{n-1}{c_{1-\alpha/2}} s^2 \right]$$

- $t$ and $\chi^2$ have $n - 1$ degrees of freedom.
$z$ rule of thumb

Suppose $x_1, \ldots, x_n \sim N(\mu, \sigma^2)$ with $\sigma$ known.

The rule-of-thumb 95% confidence interval for $\mu$ is:

$$\left[ \bar{x} - 2 \frac{\sigma}{\sqrt{n}}, \quad \bar{x} + 2 \frac{\sigma}{\sqrt{n}} \right]$$

A more precise 95% confidence interval for $\mu$ is:

$$\left[ \bar{x} - 1.96 \frac{\sigma}{\sqrt{n}}, \quad \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}} \right]$$
Board question: computing confidence intervals

The data 4, 1, 2, 3 is drawn from $\text{N}(\mu, \sigma^2)$ with $\mu$ unknown.

(a) Find a 90% $z$ confidence interval for $\mu$, given that $\sigma = 2$.

For the remaining parts, suppose $\sigma$ is unknown.

(b) Find a 90% $t$ confidence interval for $\mu$.

(c) Find a 90% $\chi^2$ confidence interval for $\sigma^2$.

(d) Find a 90% $\chi^2$ confidence interval for $\sigma$.

(e) Given a normal sample with $n = 100$, $\bar{x} = 12$, and $s = 5$, find the rule-of-thumb 95% confidence interval for $\mu$. 
Conceptual view of confidence intervals

- Computed from data ⇒ **interval statistic**
- ‘Estimates’ a parameter of interest ⇒ **interval estimate**
- Width = measure of precision
- Confidence level = measure of performance
- Connected to NHST: Given a test statistic $x$, the confidence interval for $\theta = \theta_0$ for which we wouldn’t reject the null hypothesis $\theta = \theta_0$.
- Confidence intervals are a frequentist method.
  - No need for a prior, only uses likelihood.
  - Frequentists do not assign probabilities to hypothetical values of unknown parameters.
  - A 95% confidence interval of $[1.1, 3.3]$ for $\mu$ does not mean that $P(1.1 \leq \mu \leq 3.3) = 0.95$.
- We will compare with Bayesian probability intervals later.

Applet:
https://mathlets.org/mathlets/confidence-intervals/
Discussion: Width of confidence intervals

The quantities \( n, c = \text{confidence}, \bar{x}, \sigma \) all appear in the \( z \) confidence interval for the mean.

How does the width of a confidence interval for the mean change if:

1. We increase \( n \) and leave the others unchanged?
2. We increase \( c \) and leave the others unchanged?
3. We increase \( \mu \) and leave the others unchanged?
4. We increase \( \sigma \) and leave the others unchanged?

(A) it gets wider (B) it gets narrower (C) it stays the same.
Intervals and pivoting

\( \bar{x} \): sample mean (statistic)

\( \mu_0 \): hypothesized mean (not known)

\[ \mu_0 \pm 1 \]

interval centered on \( \mu_0 \) does not contain \( \bar{x} \)

\[ \bar{x} \pm 1 \]

interval centered on \( \bar{x} \) does not contain \( \mu_0 \)

\[ \mu_0 \pm 2.3 \]

interval centered on \( \mu_0 \) contains \( \bar{x} \)

\[ \bar{x} \pm 2.3 \]

interval centered on \( \bar{x} \) contains \( \mu_0 \)

Algebra of pivoting:

\[ \mu_0 - 2.3 \leq \bar{x} \leq \mu_0 + 2.3 \]

\( \bar{x} \) in non-rejection region

\[ \bar{x} + 2.3 \geq \mu_0 \geq \bar{x} - 2.3 \]

\( \mu_0 \) in confidence interval

\[ |\bar{x} - \mu_0| \leq 2.3 \]

distance apart \( \leq 2.3 \).

(Can also do for non-symmetric intervals.)
Board question: confidence intervals and non-rejection regions

Suppose \( x_1, \ldots, x_n \sim N(\mu, \sigma^2) \) with \( \sigma \) known.

Consider two intervals:

1. The \( z \) confidence interval around \( \bar{x} \) at confidence level \( 1 - \alpha \).
2. The \( z \) non-rejection region for \( H_0 : \mu = \mu_0 \) at significance level \( \alpha \).

Compute and sketch these intervals to show that:

\[
\mu_0 \text{ is in the first interval} \iff \bar{x} \text{ is in the second interval}.
\]
Solution

Confidence interval: \[ \bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \]

Non-rejection region: \[ \mu_0 \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \]

Since the intervals are the same width they either both contain the other’s center or neither one does.
Polling: a binomial proportion confidence interval

Data $x_1, \ldots, x_n$ from a Bernoulli($\theta$) distribution with $\theta$ unknown.

A conservative normal† $(1 - \alpha)$ confidence interval for $\theta$ is given by

$$\left[ \bar{x} - \frac{z_{\alpha/2}}{2\sqrt{n}}, \quad \bar{x} + \frac{z_{\alpha/2}}{2\sqrt{n}} \right].$$

Proof uses the CLT and the observation $\sigma = \sqrt{\theta(1 - \theta)} \leq 1/2$.

Political polls often give a margin-of-error of $\pm 1/\sqrt{n}$. This rule-of-thumb corresponds to a 95% confidence interval:

$$\left[ \bar{x} - \frac{1}{\sqrt{n}}, \quad \bar{x} + \frac{1}{\sqrt{n}} \right].$$

(The proof is in the class 23 notes.)

Conversely, a margin of error of $\pm 0.05$ means 400 people were polled.

†There are many types of binomial proportion confidence intervals.

Board question: Polling

For a poll to find the proportion $\theta$ of people supporting X we know that a $(1 - \alpha)$ confidence interval for $\theta$ is given by

$$
\left[ \bar{x} - \frac{z_{\alpha/2}}{2\sqrt{n}}, \quad \bar{x} + \frac{z_{\alpha/2}}{2\sqrt{n}} \right].
$$

(a) How many people would you have to poll to have a margin of error of 0.01 with 95% confidence? (You can do this in your head.)

(b) How many people would you have to poll to have a margin of error of 0.01 with 80% confidence. (You’ll want R or other calculator here.)

(c) If $n = 900$, compute the 95% and 80% confidence intervals for $\theta$. 