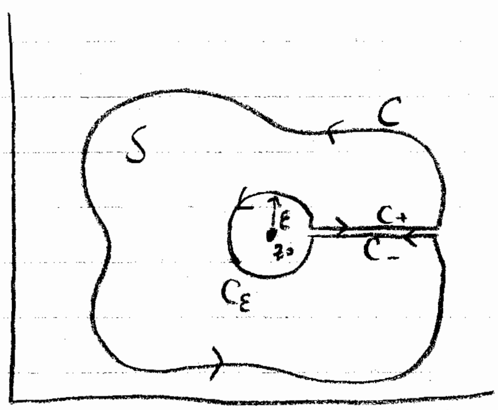
ML-formula $|f(z)| \le M$, z in S

$$\left| \int_C f(z)\,dz \right| \le M \cdot (\text{length of } C)$$

Cauchy Integral Formula: if $f(z)$ is analytic,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) \qquad \left(\text{or } f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\right)$$

Proof:

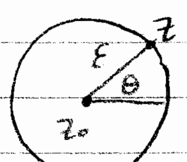


$$\tilde{C} = C + C_\varepsilon + \cancel{C_+ + C_-}$$

$$\oint_{\tilde{C}} \frac{f(z)}{z-z_0}\,dz = 0$$

$$\rightarrow \oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C_\varepsilon} \frac{f(z)}{z-z_0}\,dz$$

$$\oint_{C_\varepsilon} \frac{f(z)}{z-z_0}\,dz = \int \frac{f(z_0+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}(\varepsilon i e^{i\theta}\,d\theta) = i\int_0^{2\pi} f(z_0+\varepsilon e^{i\theta})\,d\theta$$



$z - z_0 = \varepsilon e^{i\theta}$

$dz = \varepsilon i e^{i\theta} d\theta$

$\varepsilon \to 0$: $= \boxed{i\,2\pi f(z_0)}$ ✓

Analytic Functions

$w_1 = f_1(z)$, $w_2 = f_2(z)$: analytic in region S. Then:

(i) $w_1 + w_2$: analytic in S

(ii) $w_1 w_2$: analytic in S

(iii) w_1/w_2: analytic in S at points where $w_2 \ne 0$

(iv) $\underbrace{w_1(w_2(z))}_{\text{composite function}}$: analytic for z in S' such that $w_2(z)$ is in S