18.075 Solutions to In-Class Exam #2

(I.) $$I = \int_{-\infty}^{\infty} \frac{\cos x}{(4x^2-9\pi^2)(x^2+9)}\, dx$$

(1.)
$$\text{Integrand} = \frac{\cos z}{(4z^2-9\pi^2)(z^2+9)}$$

The denominator vanishes at $4z^2-9\pi^2=0 \Leftrightarrow z=\pm\frac{3\pi}{2}$ and at $z^2+9=0 \Leftrightarrow z=\pm 3i$

The zeros at $z=\pm 3\pi/2$ are cancelled by $\cos z$.

Hence, the integrand has two simple poles at $z=\pm 3i$.

(2.) $\dfrac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)}$ has simple poles at $z=\pm\frac{3\pi}{2}, \pm 3i$.

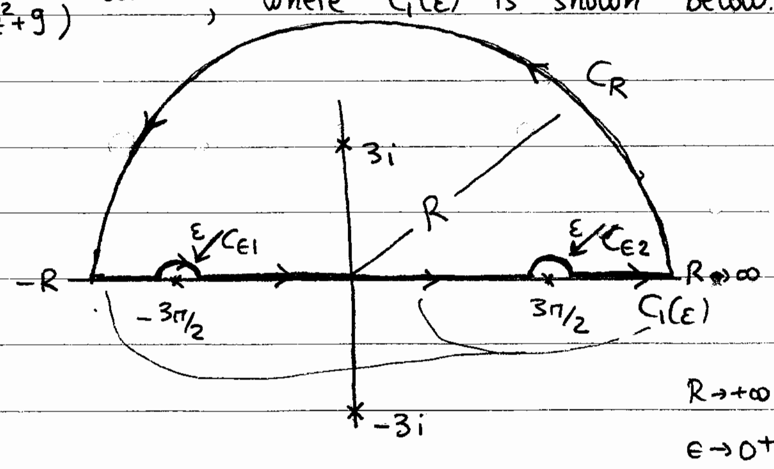
(3.)
$$I = \text{Re}\, \underbrace{P}_{\text{principal value}} \int_{-\infty}^{\infty} \frac{e^{ix}}{(4x^2-9\pi^2)(x^2+9)}\, dx \overset{\text{def.}}{\equiv} \text{Re}\left\{ \lim_{\epsilon\to 0^+} \left[\int_{-\infty}^{-\frac{3\pi}{2}-\epsilon} + \int_{-3\pi/2+\epsilon}^{3\pi/2-\epsilon} + \int_{3\pi/2+\epsilon}^{+\infty} \right] \frac{e^{ix}}{(4x^2-9\pi^2)(x^2+9)}\, dx \right\}$$

$$= \text{Re} \lim_{\epsilon\to 0^+} \int_{C_1(\epsilon)} \frac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)}\, dz,$$ where $C_1(\epsilon)$ is shown below.

Take

$C = C_R + C_1(\epsilon) + C_{\epsilon 1} + C_{\epsilon 2}$,

with $R\to+\infty$, $\epsilon\to 0^+$.



④ We take the large semicircle in the upper half plane because

$$\lim_{R\to+\infty}\int_{C_R} dz\, \frac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)} = 0 \text{ by Theorem 2.}$$

[Integral of type $\int_{C_R} dz\, f(z)\, e^{iaz}$, where $a=1>0$ and $f(z)=\frac{1}{(4z^2-9\pi^2)(z^2+9)}$ goes to 0 uniformly as $|z|\to\infty$.]

Residue theorem: $$\oint_C dz\, \frac{e^{iz}}{(4z^2-9\pi^2)\cdot(z^2+9)} = 2\pi i \operatorname*{Res}_{z=3i}\left[\frac{e^{iz}}{(4z^2-9\pi^2)\cdot(z^2+9)}\right]$$

$$= 2\pi i \operatorname*{Res}_{z=3i}\left[\frac{e^{iz}/(4z^2-9\pi^2)}{z^2+9}\right] = 2\pi i\, \frac{e^{-3}/(-4\cdot 9-9\pi^2)}{2\cdot 3i}$$

$$= -\frac{\pi}{27}\,\frac{e^{-3}}{\pi^2+4}.$$

• $$\lim_{\varepsilon\to 0}\int_{C_{\varepsilon 1}} dz\, \frac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)} = -\pi i \operatorname*{Res}_{z=-3\pi/2}\left[\frac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)}\right] = -\pi i \operatorname*{Res}_{z=-\frac{3\pi}{2}}\left[\frac{e^{iz}/(z^2+9)}{4z^2-9\pi^2}\right]$$

$$= -\pi i\, \frac{e^{-i3\pi/2}/(9+9\pi^2/4)}{8\cdot(-\frac{3\pi}{2})} = -\cancel{\pi} i\, \frac{i}{-12\cancel{\pi}\cdot 9\,(1+\frac{\pi^2}{4})} = \frac{-1}{27(\pi^2+4)}$$

• $$\lim_{\varepsilon\to 0}\int_{C_{\varepsilon 2}} dz\, \frac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)} = -\pi i \operatorname*{Res}_{z=3\pi/2}\left[\frac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)}\right] = -\pi i \operatorname*{Res}_{z=\frac{3\pi}{2}}\left[\frac{e^{iz}/(z^2+9)}{4z^2-9\pi^2}\right]$$

$$= -\pi i\, \frac{e^{i3\pi/2}/(\frac{9\pi^2}{4}+9)}{8.\,3\pi/2} = \frac{-1}{27(\pi^2+4)}$$

$$\oint_C dz\, \frac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)} = \left(\int_{C_{\varepsilon 1}} + \int_{C_{\varepsilon 2}} + \cancelto{0}{\int_{C_R}} + \int_{C_1(\varepsilon)}\right) dz\, \frac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)} = -\frac{\pi}{27}\,\frac{e^{-3}}{\pi^2+4}$$

$$\Rightarrow \lim_{\varepsilon\to 0^+}\int_{C_1(\varepsilon)} dz\, \frac{e^{iz}}{(4z^2-9\pi^2)(z^2+9)} = -\frac{\pi}{27}\,\frac{e^{-3}}{\pi^2+4} + \frac{2}{27(\pi^2+4)} = \frac{2-\pi e^{-3}}{27(\pi^2+4)}$$

$$\Rightarrow \boxed{I = \frac{2-\pi e^{-3}}{27(\pi^2+4)}}$$

II.

1. $$\sum_{n=0}^{\infty} \frac{2^n}{n^n} z^n \quad ; \qquad \text{let} \quad A_n(z) = \frac{2^n}{n^n} z^n .$$

Root test: $$\sqrt[n]{|A_n(z)|} = \sqrt[n]{\frac{2^n}{n^n} |z|^n} = \frac{2 \cdot |z|}{n} \xrightarrow[n \to \infty]{} 0 \cdot |z| .$$

It follows that $\lim\limits_{n\to\infty} \sqrt[n]{|A_n|} < 1$ for every finite $z \Rightarrow \boxed{R = \infty}$

2. $$\sum_{n=1}^{\infty} \frac{n(n+1)}{2^n} (z-1)^{3n} .$$

Let $A_n(z) = \dfrac{n(n+1)}{2^n} (z-1)^{3n}$.

$$L \equiv \left| \frac{A_{n+1}(z)}{A_n(z)} \right| = \left| \frac{\frac{\cancel{(n+1)}\,(n+2)}{2^{n+1}} (z-1)^{3n+3}}{\frac{n\cancel{(n+1)}}{2^n} (z-1)^{3n}} \right| = \frac{1}{2} \frac{n+2}{n} |z-1|^3 \xrightarrow[n\to\infty]{} \frac{1}{2} |z-1|^3 .$$

We need $L < 1$ in order for the series to converge and $L > 1$ in order for series to diverge.

Hence, the series converges for $|z-1|^3 < 2 \iff |z-1| < \sqrt[3]{2}$

and the series diverges for $|z-1|^3 > 2 \iff |z-1| > \sqrt[3]{2}$.

$$\Rightarrow \boxed{R = \sqrt[3]{2}}$$

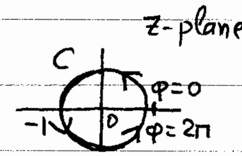
(III) $$I = \int_0^{\pi} d\theta \, \frac{1}{2+\sin^2\theta}$$

(1) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$: $$I = \int_0^{\pi} d\theta \, \frac{1}{2 + \frac{1-\cos 2\theta}{2}} = \int_0^{\pi} \frac{2\,d\theta}{5-\cos 2\theta}$$

Make the change of variable $2\theta = \varphi$: $$I = \int_0^{2\pi} \frac{d\varphi}{5-\cos\varphi}$$

Then set $z = e^{i\varphi} \Rightarrow \cos\varphi = \frac{1}{2}(z+z^{-1})$ so that

$[0, 2\pi) \longrightarrow C$ (unit circle)



(2) $$I = \oint_C \frac{dz}{iz} \frac{1}{5 - \frac{z+z^{-1}}{2}} = \oint \frac{dz}{iz} \frac{2z}{10z - z^2 - 1} = -\frac{2}{i} \oint dz \, \frac{1}{z^2 - 10z + 1}$$

Roots of denominator in integrand: $z^2 - 10z + 1 = 0 \Leftrightarrow z_{\pm} = 5 \pm \sqrt{24}$,

$z_- = 5 - \sqrt{24}$ is inside the unit circle

and $z_+ = 5 + \sqrt{24}$ is outside the unit circle.

These $z_{\pm}$ are simple poles of the integrand.

Two ways to calculate I :

(i) $$I = -\frac{2}{i} 2\pi i \cdot \operatorname*{Res}_{z=z_-} \left(\frac{1}{z^2 - 10z + 1} \right) = \frac{-2}{i} 2\pi i \, \frac{1}{2z_- - 10}$$

$$= -4\pi/2 \, \frac{1}{5 - \sqrt{24} - 5} = \frac{2\pi}{\sqrt{24}} = \frac{\pi}{\sqrt{6}},$$

by deforming the contour inside the unit circle

or (ii)

$$I = -\frac{2}{i} (-2\pi i) \operatorname*{Res}_{z=z_+} \left(\frac{1}{z^2 - 10z + 1} \right) = \frac{+2}{2i} 2\pi i \, \frac{1}{z_+ - 5}$$

$$= 2\pi \frac{1}{\sqrt{24}} = \pi/\sqrt{6},$$

by deforming the contour toward outside the unit circle

IV. $(x^2-x)y'' - (x^2+1)y' - (x-1)y = 0$.

1. By dividing both sides of this equation by x^2-x we get

$$y'' - \frac{x^2+1}{x^2-x}y' - \frac{x-1}{x^2-x}y = 0.$$

So, $a_1(x) = -\dfrac{x^2+1}{x(x-1)}$: not analytic at $x=0,1$.

$a_2(x) = -\dfrac{x-1}{x(x-1)} = -\dfrac{1}{x}$: not analytic at $x=0$.

Hence, singular points of this ode are $\underline{x_0 = 0, 1}$.

2. Take $x_0 = 0$: $(x-x_0)a_1(x) = xa_1(x) = -\dfrac{x^2+1}{x-1}$: analytic at $x_0 = 0$.

$(x-x_0)^2 a_2(x) = x^2 a_2(x) = -x$: analytic at $x_0 = 0$.

Hence, $x_0 = 0$ is a regular singular point.

3. Let $y = \sum_{n=0}^{\infty} A_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} nA_n x^{n-1}$, $y''(x) = \sum_{n=2}^{\infty} n(n-1)A_n x^{n-2}$.

$$(x^2-x)y''(x) = \sum_{n=2}^{\infty} n(n-1)A_n x^n - \sum_{n=2}^{\infty} n(n-1)A_n x^{n-1} = \sum_{n=0}^{\infty} n(n-1)A_n x^n - \sum_{n=0}^{\infty}(n+1)nA_{n+1}x^n$$

$$= \sum_{n=0}^{\infty}\left[n(n-1)A_n - (n+1)nA_{n+1}\right]x^n,$$

$$(x^2+1)y'(x) = \sum_{n=1}^{\infty} nA_n x^{n+1} + \sum_{n=1}^{\infty} nA_n x^{n-1}$$

$$= \sum_{n=2}^{\infty}(n-1)A_{n-1}x^n + \sum_{n=0}^{\infty}(n+1)A_{n+1}x^n = \sum_{n=0}^{\infty}\left[(n-1)A_{n-1} + (n+1)A_{n+1}\right]x^n,$$

by taking $\boxed{A_{-1} = 0}$

$$(x-1)y = \sum_{n=0}^{\infty} A_n x^{n+1} - \sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty} A_{n-1} x^n - \sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty} (A_{n-1} - A_n) x^n, \quad \boxed{A_{-1} = 0}.$$

Finally, the ode becomes

$$\sum_{n=0}^{\infty} \left[n(n-1) A_n - (n+1)n A_{n+1} - (n-1) A_{n-1} - (n+1) A_{n+1} - A_{n-1} + A_n \right] x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \left[(n^2-n+1) A_n - (n+1)^2 A_{n+1} - n A_{n-1} \right] x^n = 0.$$

4. From the last equation we get the recurrence formula

$$(n^2-n+1) A_n - (n+1)^2 A_{n+1} - n A_{n-1} = 0, \qquad n = 0, 1, 2, \dots$$

$\underline{n=0}$: $\quad A_0 - A_1 = 0 \Rightarrow A_1 = A_0$

$\underline{n=1}$: $\quad A_1 - 4A_2 - A_0 = 0 \Rightarrow A_2 = 0$

$\underline{n=2}$: $\quad 3A_2 - 9A_3 - 2A_1 = 0 \Rightarrow A_3 = -\frac{2}{9} A_0$

$\underline{n=3}$: $\quad 7A_3 - 16A_4 - 3A_2 = 0 \Rightarrow A_4 = \frac{7}{16} A_3 = -\frac{7}{72} A_0$, etc

So, all coefficients A_n can be expressed in terms of A_0, which is arbitrary.
Hence, this method gives only $\underline{\underline{1}}$ solution (non-trivial).

$$y(x) = A_0 \left[1 + x - \frac{2}{9} x^3 - \frac{7}{72} x^4 + \dots \right]$$

(5) $R(x)y'' + \frac{P(x)}{x} y' + \frac{Q(x)}{x^2} y = 0$, $R(0) = 1$

Original ODE: $(x^2 - x)y'' - (x^2+1) y' - (x-1)y = 0.$

Divide by $-x$: $(1-x)y'' + \frac{1+x^2}{x} y' - \frac{x-x^2}{x^2} y = 0.$

So, $R(x) = 1-x$, $P(x) = 1+x^2$, $Q(x) = -x+x^2$.

(6) $f(s) = s(s-1) + P_0 s + Q_0$, $P_0 = 1$, $Q_0 = 0$.

$f(s) = s(s-1) + s = s^2$

Indicial equation: $f(s) = 0 \Leftrightarrow s^2 = 0 \Leftrightarrow s = 0$ (double root)

So, the Frobenius method will only give 1 solution in form $y(x) = x^s \sum_{n=0}^{\infty} A_n x^n$.