Sturm-Liouville Problem

"standard" form of ODE

$$\begin{cases} \text{ODE: } \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right]y = 0, \quad a < x < b \\ \quad + \text{ homogenous boundary conditions for } y(x), \text{ at } x = a, b \end{cases}$$

Any 2nd order linear ODE of form

$$a_0(x)y'' + a_1(x)y' + [a_2(x) + \lambda\, a_3(x)]y = 0$$

can be put in standard form.

$$py'' + p'y' + (q + \lambda r)y = 0$$

$$y'' + \frac{p'}{p}y' + \frac{q + \lambda r}{p}y = 0 \qquad p \neq 0$$

$$y'' + \frac{a_1(x)}{a_0(x)}y' + \frac{a_2(x) + \lambda\, a_3(x)}{a_0(x)}y = 0$$

$$\frac{a_1}{a_0} = \frac{p'}{p} = \frac{d}{dx}\ln p \leftrightarrow \ln p = \int \frac{a_1(x)}{a_0(x)}dx \longleftrightarrow \boxed{p(x) = \exp\left(\int \frac{a_1(x)}{a_0(x)}dx\right)}$$

$$\boxed{r(x) = \frac{a_3(x)}{a_0(x)}p(x)} \qquad \boxed{q(x) = p(x)\frac{a_2(x)}{a_0(x)}}$$

Statements for S-L problem

1_ If S-L problem is satisfied by $\lambda = \lambda_m$ and by $\lambda = \lambda_n$ with solutions $\underset{(\lambda=\lambda_m)}{y = \phi_m}$ and $\underset{(\lambda=\lambda_n)}{y = \phi_n}$ then $\int_a^b dx\, r(x)\, \phi_m(x)\, \phi_n(x) = 0$ if $\lambda_n \neq \lambda_m$.

weight function

Definition: Proper S-L problem: homogeneous conditions are

$$y(a) + \gamma_a y'(a) = 0$$
$$y(b) + \gamma_b y'(b) = 0$$

constants

- $\gamma_a < 0$, $\gamma_b > 0$
- $p(x) > 0$, $r(x) > 0$, $q(x) \leq 0$ all x in (a,b)

Theorem: For a proper S-L problem, λ in $\{\lambda_n\}_{n=1,2,\ldots}$ discrete set and y in $\{\phi_n(x)\}$.

$\{\phi_n\}$ is orthogonal, ie. $\int dx\, r(x) \phi_n \cdot \phi_m = 0$ if $\lambda_n \neq \lambda_m$

$\{\phi_n\}$ can be chosen to be orthonormal, $\int_a^b dx\, r(x) \phi_n^2 \equiv 1$