ex find Laurent series: $f(z) = e^{1/z}$ $\quad z_0 = 0$

$$e^w = 1 + w + \frac{w^2}{2!} + \dots + \frac{w^n}{n!} \rightarrow e^{\frac{1}{z}} = \frac{1}{z^n n!} \quad \text{Laurent series}$$

$\frac{1}{z^n} \to \infty$, $z \to 0$, involves non positive powers of $(z - z_0)$, so series converges for $|z| > 0$.

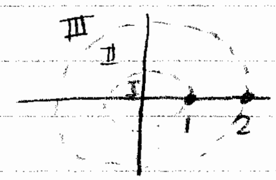
ex find Laurent series: $\frac{\sin z}{z^2}$, $\quad z_0 = 0$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

$$f(z) = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} \dots \quad \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n+1)!} \quad \text{Converges for } |z| > 0$$

ex $f(z) = \frac{1}{z^2 - 3z + 2}$ at $z_0 = 0$

$$= \frac{1}{(z-2)(z-1)}$$



a) identify singular points
b) separation of plane to annuli
c) find Laurent series in regions separately.

a, b) $f(z) = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow A = -1, \; B = 1$

$$= \underbrace{\frac{-1}{z-1}}_{f_1(z)} + \underbrace{\frac{1}{z-2}}_{f_2(z)}$$

I $|z| < 1$ $\quad f_1(z) = \frac{-1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1$

$$f_2(z) = \frac{1}{z-2} = \left(-\frac{1}{2}\right) \frac{1}{1 - \frac{z}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

true for I and II

$$\left|\frac{z}{2}\right| < 1 \leftrightarrow |z| < 2$$

$$f(z) = f_1(z) + f_2(z) = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^n, \quad |z| < 1$$

(Taylor series)

II $1<|z|<2$

$$f_1(z)=\frac{-1}{z-1}=\frac{-1}{z}\left(\frac{1}{1-1/z}\right)=\left(\frac{-1}{z}\right)\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$\left|\frac{1}{z}\right|<1\rightarrow|z|<1$$

$$f_2(z)=\frac{1}{z-2}=\text{same as I}=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$f(z)=f_1(z)+f_2(z)=-\sum_{n=0}^{\infty}z^{-n-1}-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}},\quad 1<|z|<2$$

(Laurent series)

III $|z|>2$

$$f_1(z)=\text{Same as for II}=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\qquad |z|>1$$

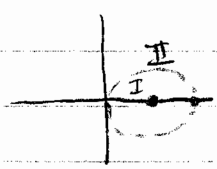
$$f_2(z)=\frac{1}{z-2}=\left(\frac{1}{z}\right)\frac{1}{1-2/z}=\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n=\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

$$\left|\frac{2}{z}\right|<1,\quad |z|>2$$

$$f(z)=f_1(z)+f_2(z)=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}=\sum_{n=1}^{\infty}(2^n-1)\frac{1}{z^{n+1}}$$

(Laurent series)

ex find $f(z)=\frac{1}{(z-1)(z-2)}$ around $z_0=1$, $\underbrace{|z-1|<1}_{\mathrm{I}}$. let $z-z_0=w$



$$f(z)=\frac{1}{(z-1)(z-2)}=\frac{1}{w}\cdot\frac{1}{w-1},\quad w<1$$

$$=\frac{1}{w(-1)}\frac{1}{1-w}=-\frac{1}{w}\sum_{n=0}^{\infty}w^n=-\sum_{n=0}^{\infty}w^{n-1}=\boxed{-\sum_{n=0}^{\infty}(z-1)^{n-1}}$$