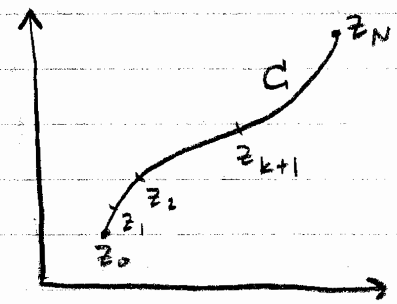
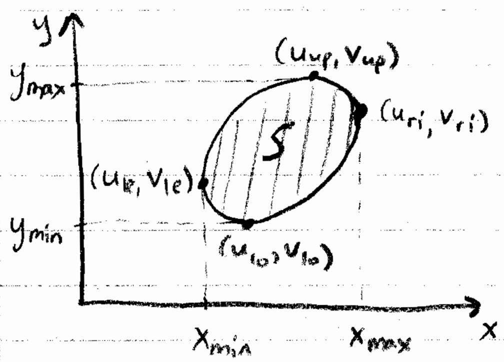
Complex Integrals



define $\int_C f(z)dz = \sum_{k=0}^{N-1} f(z_k)\cdot\Delta z_k$

$N\to\infty$ $(\Delta z_k \to 0)$

$\boxed{\text{\underline{Cauchy Integral Theorem}: If } f(z) \text{ is analytic inside C and continuous on C, } \oint_C f(z)dz = 0}$



$f = u + iv$

$\oint f(z)dz = \oint_c (u+iv)(dx+idy) = \oint_c udx + vdy + i\oint_c udy + vdx$

$(dz = dx + idy)$

$\oint udx = \int_{x_{min}}^{x_{max}} u_{lo}dx - \int_{x_{min}}^{x_{max}} u_{up}\,dx = \int_{x_{min}}^{x_{max}} \underbrace{(u_{lo} - u_{up})}_{-\int_{y_{min}}^{y_{max}} \frac{\partial u}{\partial y}dy} dx \to -\iint_S \frac{\partial u}{\partial y} dxdy$

$\oint vdy = \int_{y_{min}}^{y_{max}} (v_{ri} - v_{le})dy = \iint_S \frac{\partial v}{\partial x} dxdy$

$\oint_c udx - \oint vdy = -\iint \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)dxdy$

$\oint udy + \oint vdx = \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy$

$\to \oint_c f(z)dz = -\iint \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)dxdy + i\iint\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy$ ← for <u>any</u> f

$$\int_C (u\,dx - v\,dy) = -\iint_S \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dx\,dy \quad ①$$

Similarly, $\int_C (u\,dy + v\,dx) = \iint_S dx\,dy \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$ ②

$$\therefore \int_C f(z)\,dz = ① + i② = -\iint \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dx\,dy + i\iint\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

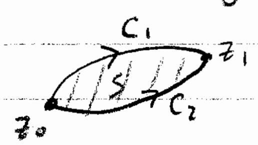
ex Find $\oint_C \bar{z}\,dz$, $\bar{z} = x+iy$

$$-\iint (0+0)\,dx\,dy + i\iint(1+1)\,dx\,dy = 2i \underbrace{\iint dx\,dy}_{\text{area enclosed by } C} = 2i\,(\text{Area})$$

Restrict to $f(z)$: analytic in $S \rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

$\rightarrow \underline{\oint f(z)\,dz = 0}$ (Cauchy integral theorem)

Suppose $f(z)$: analytic in S.



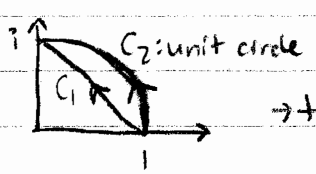
$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

independent of path

$f(z)\,dz = d\,G(z)$: exact differential of f

$$\int_{z_0}^{z_1} f(z)\,dz = \int_{z_0}^{z_1} d\,G(z) = G(z_1) - G(z_0)$$

ex $f(z) = \frac{1}{z}$



$\rightarrow$ take any path! $z = re^{i\theta}$, $r=1$

$0 \le \theta \le \frac{\pi}{2}$, $dz = ie^{i\theta}\,d\theta$

① $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz = \int_0^{\frac{\pi}{2}} ie^{i\theta}\,d\theta \left(\frac{1}{e^{i\theta}}\right) = \boxed{i\frac{\pi}{2}}$

② $\int_1^i \frac{dz}{z} = \int_1^i d\,\mathrm{Ln}\,z = \underset{\uparrow\, 1e^{i\frac{\pi}{2}}}{\mathrm{Ln}\,i} - \underset{\uparrow\, 1e^{i0}}{\overset{0}{\mathrm{Ln}(1)}} = (i\tfrac{\pi}{2}) - (i0) = \boxed{i\frac{\pi}{2}}$