18.085 Pset #1 Solutions

August Trollbäck

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Question 1.

Elimination on $\mathbf{A}_1$ (R_i represents row i):

Steps	Result
start	$\begin{pmatrix} 1 & -3 & 2 & 1 \\ 2 & -4 & 0 & -1 \\ -1 & 0 & -2 & 3 \\ 3 & -3 & 0 & -2 \end{pmatrix}$
$-2R_1 + R_2 \to R_2$ $R_1 + R_3 \to R_3$ $-3R_1 + R_4 \to R_4$	$\begin{pmatrix} 1 & -3 & 2 & 1 \\ 0 & 2 & -4 & -3 \\ 0 & -3 & 0 & 4 \\ 0 & 6 & -6 & -5 \end{pmatrix}$
$2R_3 \to R_3$	$\begin{pmatrix} 1 & -3 & 2 & 1 \\ 0 & 2 & -4 & -3 \\ 0 & -6 & 0 & 8 \\ 0 & 6 & -6 & -5 \end{pmatrix}$
$3R_2 + R_3 \to R_3$ $-3R_2 + R_4 \to R_4$	$\begin{pmatrix} 1 & -3 & 2 & 1 \\ 0 & 2 & -4 & -3 \\ 0 & 0 & -12 & -1 \\ 0 & 0 & 6 & 4 \end{pmatrix}$
$4R_4 \to R_4$	$\begin{pmatrix} 1 & -3 & 2 & 1 \\ 0 & 2 & -4 & -3 \\ 0 & 0 & -12 & -1 \\ 0 & 0 & 12 & 8 \end{pmatrix}$
$4R_3 + R_4 \to R_4$	$\begin{pmatrix} 1 & -3 & 2 & 1 \\ 0 & 2 & -4 & -3 \\ 0 & 0 & -12 & -1 \\ 0 & 0 & 0 & 7 \end{pmatrix}$

Once we reach this stage the matrix is in "echelon" form. If desired this particular matrix can further be reduced into the identity matrix, which is sometimes referred to as Gauss-Jordan elimination, whereas Gaussian elimination in this context stops once the matrix is in echelon form. It is useful to distinguish between these two processes because converting a matrix into echelon form is sufficient for computing the important *LU* decomposition.

To solve $\mathbf{A}_2\mathbf{x} = \mathbf{b}$, perform Gauss-Jordan elemination using the augmented matrix $\left(\mathbf{A}_2 \mid \mathbf{b}\right)$:

Steps	**Result**
start	$\left(\begin{array}{ccc\|c} -2 & 0 & -2 & 1 \\ 8 & 10 & -4 & 1 \\ 0 & -4 & 3 & -2 \end{array}\right)$
$4R_1 + R_2 \to R_2$	$\left(\begin{array}{ccc\|c} -2 & 0 & -2 & 1 \\ 0 & 10 & -12 & 5 \\ 0 & -4 & 3 & -2 \end{array}\right)$
$(2/5)R_2 + R_3 \to R_3$	$\left(\begin{array}{ccc\|c} -2 & 0 & -2 & 1 \\ 0 & 10 & -12 & 5 \\ 0 & 0 & -9/5 & 0 \end{array}\right)$
eliminate in 3rd col	$\left(\begin{array}{ccc\|c} -2 & 0 & 0 & 1 \\ 0 & 10 & 0 & 5 \\ 0 & 0 & -9/5 & 0 \end{array}\right)$
$-(1/2)R_1 \to R_1$ $(1/10)R_2 \to R_2$ $-(5/9)R_3 \to R_3$	$\left(\begin{array}{ccc\|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{array}\right)$

Looking at the rightmost column, the solution to $\mathbf{A}_2\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$

Question 2.

A full row or column of zeros is always an indication that a matrix is singular (non-invertible) and therefore has a null space with dimension ≥ 1. To find the column space and null space, we can perform elimination on $\mathbf{B}_1$:

Steps	**Result**
start	$\begin{pmatrix} 0 & -1 & 3 \\ 1 & 4 & -1 \\ 0 & 0 & 0 \end{pmatrix}$
$R_1 \to R_2$ $R_2 \to R_1$	$\begin{pmatrix} 1 & 4 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$
$4R_2 + R_1 \to R_1$	$\begin{pmatrix} 1 & 0 & 11 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

Then we can see that a vector $(x, y, z)^T$ is in the null space if the relations $x + 11z = 0$ and $-y + 3z = 0$ hold:

$$\begin{pmatrix} 1 & 0 & 11 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \implies \begin{cases} x + 11z = 0 \\ -y + 3z = 0 \end{cases}$$

So then any vector $(-11z, 3z, z)^T$ is in the null space of $\mathbf{B}_1$ and we can give the basis:

$$N(\mathbf{B}_1) = \text{span}\left\{ \begin{pmatrix} -11 \\ 3 \\ 1 \end{pmatrix} \right\}$$

The column space of $\mathbf{B}_1$ simply corresponds to the columns that have pivot elements, namely the first and second:

$$C(\mathbf{B}_1) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} \right\}$$

Adding the dimensions of these spaces we have $\dim N(\mathbf{B}_1) + \dim C(\mathbf{B}_1) = 3$, which does indeed match the dimension of the domain $\mathbb{R}^3$.

In the row-echelon form of $\mathbf{B}_2$, obtained with Gaussian elemination on $\mathbf{A}_1$ in the previous problem, we can see there are four pivots. Thus the dimension of the column space is four, and so the column space is all of $\mathbb{R}^4$, which is spanned by the standard basis vectors $\mathbf{e}_1 = (1,0,0,0)^T$, $\mathbf{e}_2 = (0,1,0,0)^T$, $\mathbf{e}_3 = (0,0,1,0)^T, \mathbf{e}_4 = (0,0,0,1)^T$. By rank-nullity, the dimension of null space is $4-4=0$, so its basis is the empty set $\emptyset$.

Question 3.

The characteristic polynomial for $\mathbf{C}_1$ is:

$$\det(\mathbf{C}_1 - \lambda\mathbf{I}) = \det\begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} = \lambda(3+\lambda)+2 = (\lambda+1)(\lambda+2)$$

Which has roots $\lambda_1 = -1$ and $\lambda_2 = -2$. The eigenvectors corresponding to λ_1 and λ_2 are in the null space of $\mathbf{C}_1 - \lambda_1\mathbf{I}$ and $\mathbf{C}_2 - \lambda_2\mathbf{I}$, respectively. Then for λ_1 we wish to find solutions of

$$(\mathbf{C}_1 - \lambda_1\mathbf{I})\mathbf{v}_1 = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\mathbf{v}_1 = \mathbf{0}$$

So clearly $\mathbf{v}_1 = (1,-1)^T$ is a suitable eigenvector for λ_1, as is any nonzero multiple of $\mathbf{v}_1$. Performing the same steps for $\mathbf{v}_2$, we see $\mathbf{v}_2 = (1,-2)^T$ and multiples of this are eigenvectors for λ_2.

Question 4.

- The characteristic polynomial for $\mathbf{A}_1$ is $(2-\lambda)(7-\lambda)-36 = (\lambda-11)(\lambda+2)$ so we have eigenvalues $\lambda_1 = 11, \lambda_2 = -2$.

  $\mathbf{A}_2 = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$ is a triangular matrix, so we can use the fact that the eigenvalues of a triangular matrix are simply the diagonal elements (a good property to know!): $\lambda_1 = -2$ and $\lambda_2 = -4$.

- Consider $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

- $||\mathbf{u}||_{L^2} = \sqrt{(-1)^2+1^2} = \sqrt{2}$ and $||\mathbf{v}||_{L^2} = \sqrt{1^2+(-2)^2} = \sqrt{5}$

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  - For $\mathbf{A}_1$ we have $Q(\mathbf{u}, \mathbf{u}) = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 6 & 7 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -3$ and $Q(\mathbf{v}, \mathbf{v}) = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 6 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 6$
  - For $\mathbf{A}_2$ we have $Q(\mathbf{u}, \mathbf{u}) = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -6$ and $Q(\mathbf{v}, \mathbf{v}) = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 6 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -18$
- **Symmetric** matrices with all negative eigenvalues will have a negative quadratic form $Q(\mathbf{u}, \mathbf{u})$ for all nonzero vectors $\mathbf{u}$. If the eigenvalues have mixed signs, the quadratic form doesn't have a particular sign. This property is the definiteness of a matrix, and all negative eigenvalues imply negative definiteness, and also implies many other useful properties. One can analogously define positive definite matrices.

Question 5.

Statement 1 proof:

$$\det(\mathbf{A}) \det(\mathbf{A}^{-1}) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{I}) = 1 \implies \det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

Statement 2 proof (inductive):

$$\begin{aligned} \det(\mathbf{A}^n) = \det(\mathbf{A}\mathbf{A}^{n-1}) &= \det(\mathbf{A}) \det(\mathbf{A}^{n-1}) \\ &= \det(\mathbf{A}) \det(\mathbf{A}) \det(\mathbf{A}^{n-2}) \\ &\cdots \\ &= \det(\mathbf{A})^n \end{aligned}$$

Question 6.

Part 2:

If we have eigendecomposition $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, then we can compute $\mathbf{A}^k$ with:

$$\mathbf{A}^k = (\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1})^k = (\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}) \cdots (\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1})$$

Notice in the expansion that adjacent $\mathbf{V}$ and $\mathbf{V}^{-1}$ products cancel and the remaining $\mathbf{\Lambda}$ products can be collapsed into $\mathbf{\Lambda}^k$. For an inductive proof, we can say our base case is $\mathbf{A}^1 = \mathbf{V}\boldsymbol{\Lambda}^1\mathbf{V}^{-1}$ clearly. Then if we suppose $\mathbf{A}^{n-1} = \mathbf{V}\boldsymbol{\Lambda}^{n-1}\mathbf{V}^{-1}$ is true, then we see

$$\mathbf{A}^n = \mathbf{A}\mathbf{A}^{n-1} = (\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1})(\mathbf{V}\boldsymbol{\Lambda}^{n-1}\mathbf{V}^{-1}) = \mathbf{V}\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{n-1}\mathbf{V}^{-1} = \mathbf{V}\boldsymbol{\Lambda}^n\mathbf{V}^{-1}$$

So the equation holds for any $n \geq 0$, by induction.

Part 3: See next page for example code in Julia, or MATLAB solution code.

In [1]:
```julia
using LinearAlgebra
```

In [2]:
```julia
A = rand(4, 4)
```

Out[2]:
```
4×4 Array{Float64,2}:
 0.567917   0.574422  0.614411  0.637295
 0.760314   0.482641  0.259178  0.969808
 0.0679652  0.653934  0.386424  0.384521
 0.0170879  0.349758  0.572598  0.265557
```

In [3]:
```julia
eig = eigen(A)
V, Λ = eig.vectors, Diagonal(eig.values);
```

In [4]:
```julia
A^4
```

Out[4]:
```
4×4 Array{Float64,2}:
 2.73039  4.11606  3.69293  4.55391
 2.78521  4.18318  3.73266  4.6441
 1.70599  2.60291  2.32884  2.85865
 1.27137  1.91894  1.73656  2.11971
```

In [5]:
```julia
# real part matches A^4 to at least 6 figures
real(V * Λ^4 * inv(V))
```

Out[5]:
```
4×4 Array{Float64,2}:
 2.73039  4.11606  3.69293  4.55391
 2.78521  4.18318  3.73266  4.6441
 1.70599  2.60291  2.32884  2.85865
 1.27137  1.91894  1.73656  2.11971
```

In [6]:
```julia
# imaginary part is just roundoff errors
imag(V * Λ^4 * inv(V))
```

Out[6]:
```
4×4 Array{Float64,2}:
 -1.39689e-16  3.58586e-16  -7.75452e-16  1.06789e-16
 -1.42747e-16  3.64724e-16  -7.89298e-16  1.08895e-16
 -8.85696e-17  2.26913e-16  -4.87979e-16  6.66013e-17
 -6.44e-17     1.67358e-16  -3.61549e-16  4.96017e-17
```

In [7]:
```julia
using BenchmarkTools
A = rand(1000, 1000)
# force julia to do naive matrix power
As = repeat([A], 25)
@btime prod(As);
```

```
  224.625 ms (48 allocations: 183.11 MiB)
```

In [8]:
```julia
eig = eigen(A)
V, Λ = eig.vectors, Diagonal(eig.values)
@btime V * Λ^25 * inv(V);
```

```
  109.500 ms (24 allocations: 46.85 MiB)
```