18.085/18.0851 Computational Science and Engineering I	Summer 2020
Week 7 (July 20th-July 24th)	
Lecturer: Richard Zhang	*Scribes: Richard Zhang*

With the appropriate tools, we shall dive in to partial differential equations (PDE). Partial differential equations is a vast field research whose applications range from engineering and science to economics and finance. For this lecture, we will start with the simpliest partial differential equation: the Laplace's equation

7.1 The Laplace's Equation

The Laplace's equation is simplying saying that the Laplacian of a function u is zero, ie.

$$\Delta u = 0 \tag{7.1}$$

Here $u(x_1, ..., x_n) : \mathbb{R}^n \to \mathbb{R}$. Recall that this means

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{7.2}$$

In reality, n is at most 3 since the world is three dimensional. Some examples of Laplace's equation are

- Electrostatics: if u is the electric potential in vacuum, then it obeys $\Delta u = 0$
- Heat equation: if u is the temperature of an object, then it obeys $u_t = \Delta u$. If u does not change in time anymore, then the constant temperature would distribute itself in a way in space in a way that obeys the Laplace's equation $\Delta u = 0$
- Fluid dynamics: The velocity of an incompressible fluid, v, would obey $\nabla \cdot v = 0$. If we let $u = \nabla v$, then we can write $\Delta u = 0$, which is exactly the Laplace's equation.

Now let us first explore their analytical solution

7.2 Separation of Variables

For now we focus on $n = 2$, ie.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{7.3}$$

The first attempt to solving the Laplace's equation is separation of variables. The whole idea revolves around the assumption that we can write u as a function of just x and a function of just y, ie.

$$u(x, y) = X(x)Y(y) \tag{7.4}$$

This is a bold assumption that we do not know a-priori. But let's suppose it to be the case. Then we can plug the solution into the equation and see that

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0 \tag{7.5}$$

Now we divide both sides by XY and see that

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \tag{7.6}$$

We notice that the sum is made of two parts: the first part is independent of y and the second part is independent of x. Generally speaking, there is no functions of independent variables that add up to zero, unless each part equals a constant, ie.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = a \tag{7.7}$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = b \tag{7.8}$$

$$a + b = 0 \tag{7.9}$$

Now we must proceed with care. Which one is negative and which other one is positive depends on the boundary conditions.

7.2.1 Example of Boundary Condition 1

Let's say we are solving the $2D$ Laplace's equation over the square bounded by $y = 0$, $y = L$, $x = 0$, and $x = L$. The boundary condition is defined such that

$$\begin{cases} u(x,0) = 0 \\ u(x,L) = x \\ u(0,y) = 0 \\ u(L,y) = 0 \end{cases} \tag{7.10}$$

Then we realize that along the x direction, $u(x,y)$ is periodic, where as along the y direction, $u(x,y)$ is not periodic. Hence it makes sense that

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\lambda^2 \tag{7.11}$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \lambda^2 \tag{7.12}$$

This is because a negative number on the right-hand side would render a sinusoidal pattern, whereas a positive number would make it exponential. Henceforth,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\lambda^2 \tag{7.13}$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \lambda^2 \tag{7.14}$$

To refresh your memory, let us compute the solution to the ordinary differential equation.

Assuming $X = Ae^{bx}$. Then

- $X'(x) = Abe^{bx}$
- $X''(x) = Ab^2e^{bx}$

Hence we have that

$$Ab^2e^{bx} + \lambda^2 Ae^{bx} = 0 \tag{7.15}$$
$$b^2 + \lambda^2 = 0 \tag{7.16}$$
$$b = \pm i\lambda \tag{7.17}$$

In this way,

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x) \tag{7.18}$$

Similarly, we can compute that

$$Y(y) = Ce^{\lambda y} + De^{-\lambda y} \tag{7.19}$$

Now let's eliminate some variables.

First, $u(0, y) = X(0)Y(y) = 0$ means that $X(0) = 0$, whence

$$X(0) = A\sin(\lambda 0) + B\cos(\lambda 0) \tag{7.20}$$
$$= B \tag{7.21}$$
$$= 0 \tag{7.22}$$

Secondly, $u(L, y) = X(L)Y(y) = 0$ means that $X(L) = 0$, whence

$$X(L) = A\sin(\lambda L) \tag{7.23}$$
$$= 0 \tag{7.24}$$

This means that $\sin(\lambda L) = 0$. We know that the sine function can only be zero at multiples of π, whence

$$\lambda L = n\pi \tag{7.25}$$
$$\lambda = \frac{n\pi}{L} \tag{7.26}$$

Continuing on, we have that $u(x, 0) = X(x)Y(0) = 0$, whence $C + D = 0$.

Hence, we can write the solution to $u(x, y)$ as an infinite sum

$$u(x, y) = \sum_{n=1}^{\infty} D_n(e^{\frac{n\pi}{L}y} - e^{-\frac{n\pi}{L}y})\sin\left(\frac{n\pi x}{L}\right) \tag{7.27}$$

We plug in the last boundary condition, $u(x, L) = x$

$$u(x, L) = x \tag{7.28}$$
$$= \sum_{n=1}^{\infty} D_n(e^{n\pi} - e^{-n\pi})\sin\left(\frac{n\pi x}{L}\right) \tag{7.29}$$

Solving for D_n becomes a Fourier series problem. Using what we will explore in the homework*, we can write

$$D_n(e^{n\pi} - e^{-n\pi}) = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right) \tag{7.30}$$

Doing this integral requires integration by parts, which gives

$$D_n(e^{n\pi} - e^{-n\pi}) = \frac{2}{L}\left(-\frac{x}{\frac{n\pi}{L}}\cos\left(\frac{n\pi x}{L}\right)|_0^L - \int_0^L \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)dx\right) \tag{7.31}$$

The second term disappears due to the fact that sines vanish at multiples of π. Hence

$$D_n(e^{n\pi} - e^{-n\pi}) = -\left(\frac{2L}{n\pi}\cos(n\pi)\right) \tag{7.32}$$

$$= -\frac{2L}{n\pi}(-1)^n \tag{7.33}$$

$$= \frac{2L}{n\pi}(-1)^{n+1} \tag{7.34}$$

We can then write down the solution

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{n\pi(e^{n\pi} - e^{-n\pi})}(-1)^{n+1}(e^{n\pi y/L} - e^{-n\pi y/L})\sin\left(\frac{n\pi x}{L}\right) \tag{7.35}$$

7.3 Laplace's Equation in Polar Coordinate

For various engineering purposes, it can be much easier to solve the Laplace's equation in non-Cartesian coordinates, such as the polar coordinate in $2D$. As a reminder, a point in $\mathbb{R}^2$ can be represented either by its $x - y$ coordinate or $r - \theta$ coordinate. As a reminder, we can transform between the two coordinates as follows

$$x = r\cos\theta \tag{7.36}$$

$$y = r\sin\theta \tag{7.37}$$

$$r = = \sqrt{x^2 + y^2} \tag{7.38}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{7.39}$$

The Laplace's equation in the polar coordinate is as follows

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{7.40}$$

You can find the derivation of the Laplacian in polar coordinate in the following link:

https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates (7.41)

This contains the transform of partial derivatives and various del operators in different coordinates

7.4 Separation of Variables on Laplace Equation in Polar Coordinate

We would like to analytically solve the Laplace's equation in the polar coordinate, which looks like

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{7.42}$$

We again assume that u is separable, ie.

$$u(r,\theta) = R(r)\Theta(\theta) \tag{7.43}$$

ie. u can be separated into a product of something purely dependent on r and something purely dependent on θ. We then plug in the solution back into the equation and obtain

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{7.44}$$

$$= \frac{\Theta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} \tag{7.45}$$

$$= 0 \tag{7.46}$$

Now we divide $R\Theta$ on both sides

$$\Delta u = 0 \tag{7.47}$$

$$\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} = 0 \tag{7.48}$$

After we multiply both sides by r^2, we see that

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} = 0 \tag{7.49}$$

Which again is the sum of something independent of θ and something independent of r. Since, any functions of angles cannot be multi-valued, we must let Θ have the periodicity, whence

$$\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} = -m^2 \tag{7.50}$$

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = m^2 \tag{7.51}$$

Notice that I am using m instead of λ to suggest that it should be an integer. This is because the solution to the angular part should be $\cos(m\theta)$ and $\sin(m\theta)$ If m is not an integer, then $\Theta(0)$ would have a different value than $\Theta(2\pi)$. Hence, it must be an integer.

The second question is: what happens when m is 0? In that case, $\Theta = A_0 + B_0\theta$ and

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = 0 \tag{7.52}$$

$$r\frac{\partial R}{\partial r} = C_0 \tag{7.53}$$

$$R = C_0\ln(r) + D_0 \tag{7.54}$$

Now once again, since Θ cannot be multi-valued, $B_0 = 0$.

Next, if $m > 0$, then we have that

$$\Theta(\theta) = A_m\cos(m\theta) + B_m\sin(m\theta) \tag{7.55}$$

$$R(r) = C_m r^m + D_m r^{-m} \tag{7.56}$$

Hence, the complete solution should be

$$u(r,\theta) = A_0(C_0 \ln(r) + D_0) + \sum_{m=1}^{\infty}(A_m \cos m\theta + B_m \sin m\theta)(C_m r^m + D_m r^{-m}) \tag{7.57}$$

The coefficients will depend on the boundary condition.

7.4.1 Example of Boundary Condition

Suppose that we have a circle of radius 1 centered at the origin. We would like to solve the Laplace's equation outside the circle. At the boundary, the solution would look like

$$u(r=1,\theta) = \begin{cases} -1, -\pi \leq \theta < 0 \\ 1, 0 \leq \theta \leq \pi \end{cases} \tag{7.58}$$

First, we recall the general solution as before

$$u(r,\theta) = A_0(C_0 \ln(r) + D_0) + \sum_{m=1}^{\infty}(A_m \cos m\theta + B_m \sin m\theta)(C_m r^m + D_m r^{-m}) \tag{7.59}$$

As $r \to \infty$, $u(r,\theta)$ needs to remain finite, which means that

- $C_0 = 0$, since $\ln(r) \to \infty$ as $r \to \infty$
- $C_m = 0$ for all m, since $r^m \to \infty$ as $r \to \infty$

We are left with

$$\begin{aligned} u(r,\theta) &= A_0 + \sum_{m=1}^{\infty}(A_m \cos m\theta + B_m \sin m\theta)(D_m r^{-m}) && (7.60)\\ &= A_0 + \sum_{m=1}^{\infty}(A_m \cos m\theta + B_m \sin m\theta) r^{-m} && (7.61)\\ & && (7.62) \end{aligned}$$

where we absorb the constant D_m into A_m and B_m since both of them are arbitrary constants anyway.

Now to get A_m and B_m, we plug in $r = 1$ and get that

$$\begin{aligned} u(r=1,\theta) &= A_0 + \sum_{m=1}^{\infty}(A_m \cos m\theta + B_m \sin m\theta) && (7.63)\\ & && (7.64) \end{aligned}$$

This is nothing but Fourier series. To get the solution, we simply look up the Fourier series of the step function, which we have done last week

$$u(r=1,\theta) = \sum_{k=1,3,5,\ldots}^{\infty} \frac{4}{k\pi}\sin(k\theta) \tag{7.65}$$

Hence we see that

- $A_0 = 0$
- $A_m = 0$, for all $m = 1, 2, 3, ...$
- $B_m = 0$, for $m = 2, 4, 6, ...$
- $B_m = \frac{4}{m\pi}$

Therefore, the final solution of $u(r, \theta)$ should be

$$u(r,\theta) = \sum_{m=1,3,5,...}^{\infty} \frac{4}{m\pi} \sin(m\theta) r^{-m} \tag{7.66}$$

7.5 A Note about Indices and Variables

We have lots of indices and variables floating around, which inevitably leads to typos. It is important to know which are the indices that matter when mislabeled and which do not matter when mislabeled.

7.5.1 Dummy Indices / Variables

When we write a summation, such as

$$\sum_{k=1}^{\infty} a_k \tag{7.67}$$

Here k is called a dummy variable. It is simply a notation and a placeholder for the summation. Hence, it does not matter whether I call it k or n, as either will be summed eventually from 1 to ∞.

Similarly, when we write an integral

$$\int f(t) dt \tag{7.68}$$

Here t is called a dummy variable. I could have used x and it does not make a difference.

7.5.2 Non-Dummy Indices Variables

When I isolate individual indices or variables, they become non-dummmy. For instance, suppose I have a Fourier series

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(kx) \tag{7.69}$$

And I want to use the orthogonality argument to find an expression for a_n, where $n = 1, 2, 3....$ Here n is not a dummy variable. So it is important that I use n to imply the fact that it is difference from k, ie

$$\int_{-\pi}^{\pi} f(x) \sin(nx) dx = \sum_{k=1}^{\infty} a_k \int_{-\pi}^{\pi} \sin(kx) \sin(nx) dx \tag{7.70}$$

7.6 Numerical Solution to Laplace's Equation: Finite Difference

Having learning how to solve Laplace's equation analytically, in two coordinate systems, we will now focus on numerical solution.

7.6.1 Review of $1D$ Laplace's Equation Using Finite Difference

Recall what we did in $1D$: we use the second-difference approximation to approximate the second derivative, ie.

$$-\frac{d^2u}{dx^2}(x) \approx \frac{-u(x-h)+2u(x)-u(x+h)}{h^2} \tag{7.71}$$

Hence, over the interval $[0,1]$, if we have a grid of spacing equal to $h = \frac{1}{N+1}$, with $x_0 = 0$ and $x_{N+1} = 1$, we can write, for each $x_i = ih$,

$$\begin{aligned} -\frac{d^2u}{dx^2}(x_i) &\approx \frac{-u(x_i-h)+2u(x_i)-u(x_i+h)}{h^2} && (7.72)\\ &= \frac{-u(x_{i-1})+2u(x_i)-u(x_{i+1})}{h^2} && (7.73)\\ & && (7.74) \end{aligned}$$

And if we let $u(0) = a$ and $u(1) = b$, we have,

$$\begin{cases} \frac{-a+2u(x_1)-u(x_2)}{h^2} = 0 \\ \dots \\ \frac{-u(x_{i-1})+2u(x_i)-u(x_{i+1})}{h^2} = 0, i = 2,3,...,N-1 \\ \dots \\ \frac{-u(x_{N-1})+2u(x_N)-b}{h^2} = 0 \end{cases} \tag{7.75}$$

which, after some algebraic manipulation, becomes

$$\begin{cases} \frac{2u(x_1)-u(x_2)}{h^2} = \frac{a}{h^2} \\ \dots \\ \frac{-u(x_{i-1})+2u(x_i)-u(x_{i+1})}{h^2} = 0, i = 2,3,...,N-1 \\ \dots \\ \frac{-u(x_{N-1})+2u(x_N)}{h^2} = \frac{b}{h^2} \end{cases} \tag{7.76}$$

This can be case into a matrix form of $Au = b$, where $A \in \mathbb{R}^{N\times N}$, $u \in \mathbb{R}^N$, and $b \in \mathbb{R}^N$, such that

$$A_{ij} = \frac{1}{h^2}\begin{cases} 2, & i = j \\ -1, & |i-j| = 1 \\ 0, & \text{otherwise} \end{cases}, \tag{7.77}$$

$$u = [u(x_1), ..., u(x_N)]^T \tag{7.78}$$

$$b = \left(\frac{a}{h^2}, 0, ..., 0, \frac{b}{h^2}\right)^T \tag{7.79}$$

Then solving the linear system $Ax = b$, using, for example, the MATLAB command, $A\backslash b$, would give the vector u, whose i^{th} component is the approximate value of the function u at point x_i

7.6.2 Solving $2D$ Laplace's Equation Using Finite Difference

Using the finite difference scheme to discretize the Laplace's equation in $2D$ is a natural extension from the $1D$ case. Recall that in the Cartesian coordinate,

$$\Delta u(x) = \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} \tag{7.80}$$

Hence, to discretize the Laplacian operator, we would need to discretize the partial derivatives with respect to both x and y. Again recall that

$$\frac{\partial^2 u(x,y)}{\partial x^2} = \lim_{\Delta x \to \infty} \frac{u(x-\Delta x, y) - 2u(x,y) + u(x+\Delta x, y)}{\Delta x^2} \tag{7.81}$$
$$\frac{\partial^2 u(x,y)}{\partial y^2} = \lim_{\Delta y \to \infty} \frac{u(x, y-\Delta y) - 2u(x,y) + u(x, y+\Delta y)}{\Delta y^2} \tag{7.82}$$
$$\tag{7.83}$$

Henceforth, the Laplacian can be approximated as

$$-\Delta u \approx \frac{-u(x-\Delta x, y) + 2u(x,y) - u(x+\Delta x, y)}{\Delta x^2} + \frac{-u(x, y-\Delta y) + 2u(x,y) - u(x, y+\Delta y)}{\Delta y^2} \tag{7.84}$$
$$\tag{7.85}$$

As it turns out, it is slightly more convenient to deal with $-\Delta$ rather than Δ. Assuming that $\Delta x = \Delta y = h$, we continue our calculations

$$-\Delta u(x,y) \approx \frac{-u(x-h,y) - u(x+h,y) - u(x,y-h) - u(x,y+h) + 4u(x,y)}{h^2} \tag{7.86}$$
$$\tag{7.87}$$

Now imagine that we divide up our domain into a rectangular meshgrid of size h. We let $x_{ij} = (x_i, y_j)$ represent the grid point in the i^{th} row and j^{th} column. Then the value of the Laplacian of u is

$$-\Delta u(x_i, y_j) \approx \frac{-u(x_i-h, y_j) - u(x_i+h, y_j) - u(x_i, y_j-h) - u(x_i, y_j+h) + 4u(x_i, y_j)}{h^2} \tag{7.88}$$
$$= \frac{-u(x_i-h, y_j) - u(x_i+h, y_j) - u(x_i, y_j-h) - u(x_i, y_j+h) + 4u(x_i, y_j)}{h^2} \tag{7.89}$$

But since $x_i - h = x_{i-1}$, $x_i + h = x_{i+1}$, $y_j + h = y_{j+1}$ and $y_j - h = y_{j-1}$, we can continue as

$$-\Delta u(x_i, y_j) = \frac{-u(x_{i-1}, y_j) - u(x_{i+1}, y_j) - u(x_i, y_{j-1}) - u(x_i, y_{j+1}) + 4u(x_i, y_j)}{h^2} \tag{7.90}$$

Lastly, if we define $u(x_i, y_j) = u_{i,j}$, we have that

$$-\Delta u(x_i, y_j) = \frac{-u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} + 4u_{i,j}}{h^2} \tag{7.91}$$

Essentially, we stand at (x_i, y_j) as the center position (C) and examine the value of $u_{i,j}$ there, before looking to the West (W) for $u_{i-1,j}$, East (E) for $u_{i+1,j}$, North (N) for $u_{i,j+1}$, and South (S) $u_{i,j-1}$. We can then write the approximate value of the Laplacian as

$$-\Delta u(x_i, y_j) \approx \frac{-u_W - u_E - u_S - u_N + 4u_C}{h^2} \tag{7.92}$$

7.6.3 Matrix Assembly

Suppose we are solving the Laplace's equation over $[0,1]\times[0,1]$, subject to the boundary condition $f(x,y)$. Let's set up the numeric problem as follows:

- We shall divide up the x and y dimension into $N+1$ pieces. Hence, N^2 is the number of inner nodes (nodes that are not on the boundary).
- We let $h = 1/(N+1)$. h is called the mesh size
- we shall label the nodes consecutively and from bottom left to top right, $1, 2, ..., N^2$. These are called the global index
- We shall also label the nodes in the coordinate fashion, so that the most bottom left index is called $(1,1)$ and the most top right index is called (N^2, N^2). We shall call them the coordinate index

Let i,j run through all the indices of the inner nodes, ie. $i,j = 1,2,3.,,,N^2$. Our goal is to calculated $u_{i,j} = u(x_i, y_j) = u(ih, jh)$, which, according to our derivation above, satisfies

$$\frac{-u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} + 4u_{i,j}}{h^2} = 0 \tag{7.93}$$

We would like to convert the above expression into the matrix form.

7.6.3.1 Global - Coordinate Index Transformation

The form above is, unfortunately, in the coordinate index form. Since matrices are $2D$, we would really need a $3D$ matrix (aka. tensor) to fully resolve the system in the coordinate index form. Therefore, we would need to convert it into the global index before transforming back.

Let i,j be the coordinate index of a point, and let l be the global coordinate. To go from coordinate to global, we will do

$$l = (j-1) * N + (i-1) + 1 \tag{7.94}$$

$$\tag{7.95}$$

To go from global to coordinate, we will do

$$i = \mod(l-1, N) + 1 \tag{7.96}$$

$$j = (l-i)/N + 1 \tag{7.97}$$

7.6.3.2 Example of Matrix Assembly: N= 3

Suppose now that $N = 3$. We shall see how the matrix assembly goes.

First, the inner nodes are

- $\vec{x}_1 = \vec{x}_{1,1} = (x_1, y_1) = (1/3, 1/3)$
- $\vec{x}_2 = \vec{x}_{2,1} = (x_2, y_1) = (1/3, 2/3)$
- $\vec{x}_3 = \vec{x}_{1,2} = (x_1, y_2) = (1/3, 2/3)$

- $\vec{x}_4 = \vec{x}_{2,2} = (x_2, y_2) = (2/3, 2/3)$

Therefore, we can write the expression of u at these four points

$$\frac{-u_{0,1} - u_{2,1} - u_{1,0} - u_{1,2} + 4u_{1,1}}{h^2} = 0 \tag{7.98}$$

$$\frac{-u_{1,1} - u_{3,1} - u_{2,0} - u_{2,2} + 4u_{2,1}}{h^2} = 0 \tag{7.99}$$

$$\frac{-u_{0,2} - u_{2,2} - u_{1,1} - u_{1,3} + 4u_{1,2}}{h^2} = 0 \tag{7.100}$$

$$\frac{-u_{1,2} - u_{3,2} - u_{2,1} - u_{2,3} + 4u_{2,2}}{h^2} = 0 \tag{7.101}$$

Notice, however, some of the u values are on the boundary with known boundary conditions. If we let $f_{i,j}$ be the value of f evaluated at (x_i, y_j) and move them to the right side of the equation, we have

$$\frac{-u_2 - u_3 + 4u_1}{h^2} = \frac{f_{0,1} + f_{1,0}}{h^2} \tag{7.102}$$

$$\frac{-u_1 - u_4 + 4u_2}{h^2} = \frac{f_{3,1} + f_{2,0}}{h^2} \tag{7.103}$$

$$\frac{-u_4 - u_1 + 4u_3}{h^2} = \frac{f_{1,3} + f_{0,2}}{h^2} \tag{7.104}$$

$$\frac{-u_2 - u_3 + 4u_4}{h^2} = \frac{f_{3,2} + f_{2,3}}{h^2} \tag{7.105}$$

where the left-hand size has been converted into the global index form. We can then write the equation into the following matrix form $Au = b$, where

$$A = \frac{1}{h^2}\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}, b = \frac{1}{h^2}\begin{bmatrix} f_{0,1} + f_{1,0} \\ f_{3,1} + f_{2,0} \\ f_{1,3} + f_{0,2} \\ f_{3,2} + f_{2,3} \end{bmatrix} \tag{7.106}$$

Solving the matrix equation would give $u = (u_1, u_2, u_3, u_4)^T$.

7.6.3.3 General Algorithm

Henceforth, a general algorithm for a finite difference scheme to solve the Laplace's equation with Dirichlet boundary condition is as follows

- Discretize the domain into $N \times N$ inner nodes
- Create a zero matrix $A \in \mathbb{R}^{N^2 \times N^2}$ and $b \in \mathbb{R}^{N^2}$
- Set the global index l to run through 1 to N^2 and set $A(l, l) = 4/h^2$
- Convert l to the coordinate form and study its left, right, top, and bottom neighbors.
- If any of them belongs to the boundary, add the value of the function at that point to $b(l)$ before dividing by h^2
- Otherwise, set the corresponding entry of A to be -1
- Do $A\backslash b$ to get u