18.085/18.0851 Comp. Science. Engineering I **Name:** ____________
Summer 2020
Midterm
7/10/2020
Time Limit: 90 Minutes (Self-Enforced) ____________

This exam contains 11 pages (including this cover page) and 5 questions.
Total of points is 60.

You are allowed a sheet of double-side notes. Feel free to write whatever you want. There will be no use of electronic devices

Grade Table (for instructor use only)

Question	Points	Score
1	3	
2	5	
3	10	
4	22	
5	20	
Total:	60	

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1. (3 points) Which of the following are suitable Ansatz to the harmonic oscillator,

$$m\frac{d^2u}{dt^2} + cu = 0 \tag{1}$$

where u describes the displacement from equilibrium a ball of mass m connected to a spring with Hooke's constant c. Circle all that applies

A. $u(t) = A\cos(bt) + B\sin(bt)$, with A, B and b to be solved

B. $u(t) = \exp(t)(A\cos(bt) + B\sin(bt))$, with A, B and b to be solved

C. $u(t) = \exp(-t)(A\cos(bt) + B\sin(bt))$, with A, B and b to be solved

2. (5 points) For a square matrix, $A \in \mathbb{R}^{n\times n}$, circle all choices that are equivalent to the statement: A is invertible

A. The null space of A has only the trivial element, ie. $Nul(A) = \{0\}$

B. The column space of A has dimension n

C. A has no repeat eigenvalues

D. The columns of A are linearly independent

E. For all $u \in \mathbb{R}^n$, $u^TAu > 0$

3. (10 points) Suppose a matrix $A \in \mathbb{R}^{11\times 375}$ has its singular values following the pattern of $\sigma_i = 10^{12-i}, i = 1, 2, ..., 11$. In other words, the first singular value is 10^{11}, the second is 10^{10}, the third is 10^9, etc.

   a) (5 points) What is the condition number of A?

b) (5 points) Suppose we solve for x in $Ax = b$. Two questions

- i) What are the dimensions of x and b?
- ii) Suppose we do $A\backslash b$ on MATLAB and one of the entries of x is shown on our computer screen as

$$-0.483957485768899079747486584 64 \tag{2}$$

Write down the digits you would trust from this answer and explain your reasoning.

4. (22 points) Consider the following boundary value problem over $x \in [0, 1]$

$$-u'' = f(x) \tag{3}$$

a) (8 points) Let's take $f(x) = -2\delta(x - 1/4)$ and $u(0) = u(1) = 0$. Solve $u(x)$ and write it in the form of

$$u(t) = \begin{cases} ..., & 0 \le x < \frac{1}{4} \\ ..., & \frac{1}{4} \le x \le 1 \end{cases} \tag{4}$$

b) (6 points) Sketch the solution including the values of $u(x = 1/4)$

c) (6 points) Let $f = \cos(t)$. Setup but *do not solve* the discretized problem in matrix form $Au = b$ with a grid spacing of $h = 1/4$. The solution vector u to this linear system is our approximation to $u(x)$ at the grid points.

d)(2 points + 2 bonus points) Continuing on with $f = \cos(t)$, if we plot the logarithm of the L^2 error between the analytic and numeric solution against the logarithm of the number of grid points N, we can fit the data to a line

i) (2 points) What is the slope of the line and why?

ii) (2 bonus points) If $\vec{u} = (u_1, .., u_N)$ and $\vec{v} = (v_1, ..., v_N)$ are the numeric and analytic solutions, respectively, evaluated at the grid points $(x_1, ..., x_N)$, can you write down the L^2 error?

5. (20 points) Suppose $\lambda_1 = 1$ and $\lambda_2 = 2$ are the eigenvalues of a matrix A, and $v_1^T = [1, 0]$ and $v_2^T = [1, 1]$ are the corresponding eigenvectors

   a) (4 points) Calculate the matrix A

b) (4 points) Calculate the matrix A^8, its eigenvalues and its eigenvectors.

c) (4 points) For A and A^8, determine if each matrix is positive definite, negative definite, semidefinite, or indefinite

d) (8 points) Let $u(t) = [u_1(t), u_2(t)]^T$ satisfy

$$u'(t) = Au(t) \tag{5}$$
$$u(0) = [0, 1]^T \tag{6}$$

Solve for $u(t)$