18.085/18.0851 Computational Science and Engineering I **Summer 2020**

Week 4 (June 29th-July 3rd)

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4.1 Oscillation by Newton's Law

In the previous lecture, we largely examine the behavior of spring-mass systems at static equilibrium. Now we are letting our system vibrate

4.1.1 One Mass and One Spring

We start with one mass m hanging from one spring with Hooke's constant c. The top is fixed, while the bottom is free. We let $u(t)$ be the displacement of the mass away from equilibrium. In that case, we can write down Newton's second law, which relates forces with mass times acceleration

$$m\frac{d^2u}{dt^2} + cu(t) = 0 \tag{4.1}$$
$$u(0) = 0 \tag{4.2}$$
$$u'(0) = 1 \tag{4.3}$$

We propose the "ansatz" solution:

$$u(t) = A\cos(\alpha t) + B\sin(\beta t) \tag{4.4}$$

We plug the ansatz into the differential equation

$$-\frac{c}{m}u = \frac{d^2u}{dt^2} \tag{4.5}$$
$$-\frac{c}{m}A\cos(\alpha t) - \frac{c}{m}B\sin(\beta t) = -A\alpha^2\cos(\alpha t) - B\beta^2\sin(\beta t) \tag{4.6}$$

Matching the sine and cosine terms, we will obtain

$$\alpha^2 = \frac{c}{m} \tag{4.7}$$
$$\beta^2 = \frac{c}{m} \tag{4.8}$$

Henceforth, $\alpha^2 = \beta^2 = \omega^2 = \frac{c}{m}$, whence $\alpha = \beta = \sqrt{\frac{c}{m}}$ and

$$u(t) = A\cos(\omega t) + B\sin(\omega t) \tag{4.9}$$

The constants, A and B, are determined by the initial conditions. For example, if we let $u(0) = 0$ and $u'(0) = 1$, then

$$u(0) = A \tag{4.10}$$
$$= 0 \tag{4.11}$$
$$u'(0) = B\omega\cos(\omega * 0) \tag{4.12}$$
$$= 1 \tag{4.13}$$

Hence $B = 1$ and $A = 0$, and therefore we can write the solution as

$$u(t) = \frac{1}{\omega}\sin(\omega t) \tag{4.14}$$

This tells me that the solution is oscillating forever if we give it an initial velocity of 1.

4.1.1.1 Conservation of Energy

We define to total energy as $P(u) = \frac{1}{2}m(u'(t))^2 + \frac{1}{2}c(u(t))^2$, or in other words, kinetic plus potential energy. Let's verify that in this case, the total energy is conserved

$$\begin{align} P(u) &= \frac{1}{2}m(u'(t))^2 + \frac{1}{2}c(u(t))^2 \tag{4.15}\\ &= \frac{1}{2}m\cos^2(\omega t) + \frac{1}{2}\frac{c}{\omega^2}\sin^2(\omega t) \tag{4.16}\\ &= \frac{1}{2}m\cos^2(\omega t) + \frac{1}{2}c\frac{m}{c}\sin^2(\omega t) \tag{4.17}\\ &= \frac{1}{2}m(\cos^2(\omega t) + \sin^2(\omega t)) \tag{4.18}\\ &= \frac{1}{2}m \tag{4.19} \end{align}$$

The answer makes a lot of sense: initially we give the system a "kick" by endowing it with an initial velocity of 1. At the beginning there is no potential energy. So the total energy is just $\frac{1}{2}mv^2 = \frac{1}{2}m$, and it stays that way due to the conservation of energy.

4.1.2 Line of Masses and Springs

Let's analyze the vibration of n masses and $(n+1)$ springs with fixed-fixed ends. Following the same framework as developed in the previous lecture, we define the stiffness matrix $K = A^TCA$, where $A \in \mathbb{R}^{n\times(n+1)}$ and $C \in \mathbb{R}^{n\times n}$ such that

$$A_{ij} = \begin{cases} 1, & i = j \\ -1, & i - j = 1 \\ 0, & \text{otherwise} \end{cases} \quad C_{ij} = \begin{cases} c_i, & i = j \\ 0, & \text{otherwise} \end{cases} \tag{4.20}$$

The force balance equation says that the inertial force plus the spring internal force should cancel out any applied force from the outside, that is

$$Mu'' + Ku = f(t) \tag{4.21}$$

where

$$M_{ij} = \begin{cases} m_i, & i = j \\ 0, & \text{otherwise} \end{cases} \tag{4.22}$$

We should like to solve for the displacement of each mass $u(t) = (u_1(t), ..., u_n(t))^T$ subject to an external force of $f(t) = (f_1(t), ..., f_n(t))^T$.

4.1.2.1 Zero External Force

First we will consider the system when there is no external force, ie. $f(t) = 0$. Then the equation, upon some algebraic manipulation, would look like

$$Mu'' + Ku = 0 \tag{4.23}$$
$$u'' + Gu = 0 \tag{4.24}$$

where $G = M^{-1}K$. Similar to the first order system of differential equations, we shall use eigendecomposition to solve this second order system of differential equations.

First we let $G = V\Lambda V^{-1}$, where $\Lambda = diag(\lambda_1, ..., \lambda_n)$ and V is the eigenvector matrix. We then plug the decomposition into the differential equation

$$u'' + V\Lambda V^{-1}u = 0 \tag{4.25}$$
$$(V^{-1}u)'' + \Lambda(V^{-1}u) = 0 \tag{4.26}$$

Let $\tilde{u} = V^{-1}u$. Then we have the decoupled system of differential equations

$$\tilde{u}'' + \Lambda\tilde{u} = 0 \tag{4.27}$$

Hence we know that $\tilde{u}_i'' + \lambda_i\tilde{u}_i = 0$, for each $i = 1, ..., n$, whence $\tilde{u}_i = A_i\cos(\sqrt{\lambda_i}t) + t) + B_i\sin(\sqrt{\lambda_i}t)$.

Transforming it back to u, we let $v_1, ..., v_n$ be the columns of V, aka. the eigenvectors of the matrix G and multiply $\tilde{u}$ by V to obtain

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdots \\ u_n(t) \end{bmatrix} = \sum_{i=1}^{n}(A_i\cos(\sqrt{\lambda_i}t) + B_i\sin(\sqrt{\lambda_i}t))v_i \tag{4.28}$$

Let us do an example with two identical masses $m_1 = m_2 = m$ and three identical springs $c_1 = c_2 = c$. Then, as we know from the previous lecture

$$K = c\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{4.29}$$
$$M = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.30}$$

whence

$$G = M^{-1}K \tag{4.31}$$
$$= \frac{c}{m}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{4.32}$$

Diagonalizing the G matrix, we get that

$$\lambda_1 = \frac{c}{m}, v_1 = [1, 1]^T \tag{4.33}$$
$$\lambda_2 = \frac{3c}{m}, v_2 = [1, -1]^T \tag{4.34}$$

Therefore

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \left(A_1\cos\sqrt{\frac{c}{m}}t + B_1\sin\sqrt{\frac{c}{m}t}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(A_2\cos\sqrt{\frac{3c}{m}}t + B_2\sin\sqrt{\frac{3c}{m}t}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{4.35}$$

Two comments about the result above

- λ_2 is the faster mode, indicating a more rapid oscillation. By the same token, λ_1 is the slower mode, indicating a slower oscillation

- v_1 and v_2 are known as the normal modes. $v_1 = [1,1]^T$ moves the masses in the same direction, while $v_2 = [1,-1]^T$ moves the masses in the opposite direction.

The case becomes more interesting with more masses and springs. For each eigenvector, the corresponding normal mode $u_i = (A_i \cos\sqrt{\lambda_i}t + B_i \sin\sqrt{\lambda_i}t)v_i$ is a standing wave. When we add up all the normal modes, we end up with a traveling wave. This will be further explored in the subsequent course (18.086).

4.1.2.2 Non-Zero External Force

Now suppose that we exert an external force $f(t) = (f_1(t), ..., f_n(t))^T$. More often than not, all components of f would oscillate at the same frequency ω_0. Then we can solve for the system $Ku'' + Mu = \vec{f_0}\cos\omega_0 t$, where $\vec{f_0}$ is an $n-$dimensional constant vector, ie.

$$\vec{f_0} = \begin{bmatrix} f_0 \\ f_0 \\ \vdots f_0 \end{bmatrix} \tag{4.36}$$

The solution will involve ω_0 as well as the n natural frequencies $\omega_i = \sqrt{\lambda_i}$ from the eigenvalues of $M^{-1}K$

The critical case of resonance happens ω_0 is very close to one of the natural frequencies. This is known as resonance. When we push a swing, if we want push it as high as possible, we would often go along the frequency of the spring. When we walk on a bridge, we would not want it to oscillate, and therefore a good engineer would pick ω_0^2 away from any of the λ_i. This is apparently not the case for the Millenium Bridge in London, whose ignored a sideways mode.

Let's study a simple equation $mu'' + cu = \cos\omega_0 t$, where ω_0 is very close to the natural frequency $\lambda = \sqrt{\frac{c}{m}}$. It turns out that

- When ω_0 is close to λ, $u(t) = \frac{\cos\lambda t - \cos\omega_0 t}{m(\omega_0^2 - \lambda^2)}$
- When ω_0 is equal to λ, $u(t) = \frac{t\sin\omega_0 t}{2m\omega_0}$

You will explore a bit more of the math behind the solutions in the homework.

4.2 A Quick Intro to Complex Numbers

I would like to spend some time talking about complex numbers. Complex numbers play a key role in mathematics. Moreover, they will be the central tool for our study in circuit theory, nonlinear dynamics, and later on Fourier analysis. We shall use this lecture to gain a working knowledge of complex numbers

4.2.1 Imaginary Number

Everything starts with the seemingly farcical attempt to take the square root of -1. Since there is no real number that equals $\sqrt{-1}$, we shall call this number the imaginary number. Mathematicians and scientists like to denote the imaginary number as $i = \sqrt{-1}$, while engineers prefer the notation $j = \sqrt{-1}$.

Similar to real numbers, we would like to endow the imaginary number with an axis, known as the imaginary axis, and a sign, plus and minus. The fundamental unit on this imaginary axis would be i, which can be multiplied by scalar real numbers, eg. $1.5i, 3i, -\sqrt{17}i$, etc.

Note that $i^2 = -1, i^3 = -i, i^4 = 1, etc.$

4.2.2 Complex Numbers

Now that we have defined these complex numbers, we would like to integrate and broaden our definition of numbers. We've been dealing with real numbers all the way through. Now we shall define something called complex numbers. Denoted as $\mathbb{C}$, a typical complex number $z \in \mathbb{C}$ looks like

$$z = a + ib \tag{4.37}$$

where $a \in \mathbb{R}$ is known as the real part, and $b \in \mathbb{R}$ is known as the imaginary part. If we put the real and imaginary axis at 90^o to each other, we can represent this complex number z as a vector. This vector would have a magnitude $r = \sqrt{a^2 + b^2}$ and angle $\theta = \arctan\left(\frac{b}{a}\right)$.

4.2.2.1 The Euler Representation

Notice that

$$a = r\cos(\theta)b = r\sin(\theta) \tag{4.38}$$

Hence, z can also be written as

$$z = r\cos(\theta) + ir\sin(\theta) \tag{4.39}$$

Using Taylor expansion, we can motivate the following definition of the Euler representation of the complex number z

Definition 4.1 *The Euler representation of a complex number $z \in \mathbb{C}$ is defined as*

$$z = re^{i\theta} \tag{4.40}$$

where r represents its magnitude and and θ represents the polar angle on the real-imaginary plane. This polar representation is equivalent to the Cartesian representation via

$$\begin{aligned} z &= re^{i\theta} && (4.41) \\ &= r\cos(\theta) + ir\sin(\theta) && (4.42) \end{aligned}$$

Euler's representation of complex number is very useful in proving trignometric identities. For instance, let's prove $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$. First we know that $\sin(x+y)$ is the imaginary part of $e^{i(x+y)}$, since $e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$. We can also denote it as $sin(x+y) = Im(e^{x+y})$. But then

$$
\begin{aligned}
sin(x+y) &= Im(e^{i(x+y)}) && (4.43)\\
&= Im(e^{ix}e^{iy}) && (4.44)\\
&= Im((\cos(x) + i\sin(x))(cos(y) + i\sin(y))) && (4.45)\\
&= Im(\cos(x)\cos(y) + i\sin(y)\cos(x) + i\sin(x)\cos(y) - \sin(x)\sin(y)) && (4.46)\\
&= Im((\cos(x)\cos(y) - \sin(x)\sin(y)) + i(\sin(x)\cos(y) + \sin(y)\cos(x))) && (4.47)\\
&= \sin(x)\cos(y) + \sin(y)\cos(x) && (4.48)
\end{aligned}
$$

4.2.2.2 Complex conjugate

Definition 4.2 *For a complex number $z = a + ib$, the complex conjugate, $\bar{z}$, is defined as $\bar{z} = x - iy$*

4.2.3 Functions of A Complex Variable

Almost all rules of real-variable calculus carries over to complex variable calculus. For instance, if $z = x + iy$

- $f(z) = z^2 = (x+iy)^2 = x^2 + 2(x)(iy) + (iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + i(2xy)$
- $f'(z) = 2z = 2(x+iy) = 2x + i2y$

The modulo operation, $|\cdot|^2$, computes the magnitutde of z and is defined as

Definition 4.3 *For $z = x + iy$, $|z| = \sqrt{\bar{z}\cdot z} = \sqrt{x^2 + y^2}$*

Note that $|z|^2 \neq (z)^2$.

All functions of a complex variable can be evaluated by the Taylor expansion of the function.

4.2.4 Representation of Solutions Using Complex Variables

Because of the magical Euler's formula, we can conveniently represent the solutions of differential equations using complex exponentials. Let's re-solve the harmonic oscillator problem

$$m\frac{d^2u}{dt^2} + cu(t) = 0 \tag{4.49}$$

And this time we assume that $u(t) = Ae^{ibt}$, where A, a, and b are all functions of time. Plugging it in to the differential equation, we obtain that

$$
\begin{aligned}
-mAb^2e^{ibt} + cAe^{ibt} &= 0 && (4.50)\\
-mb^2 + c &= 0 && (4.51)\\
b^2 &= \frac{c}{m} && (4.52)
\end{aligned}
$$

Hence, $b_1 = \frac{c}{m}$ and $b_2 = -\frac{c}{m}$, whence the solution can be written as

$$u(t) = A_1 e^{b_1 t} + A_2 e^{b_2 t} \tag{4.53}$$

If we subject the oscillator to an initial condition of $u(0) = 0$ and $u'(0) = 1$, we obtain that

$$A_1 + A_2 = 0 \tag{4.54}$$
$$A_1 i b_1 + A_2 i b_2 = 1 \tag{4.55}$$

And we obtain that $A_1 = \frac{1}{2i}\sqrt{m/c}$ and $A_2 = -\frac{1}{2i}\sqrt{m/c}$. Then we can write the solution as

$$u(t) = \frac{1}{2i}\sqrt{m/c}(e^{b_1 t} - e^{b_2 t}) \tag{4.56}$$
$$= \frac{1}{2i}\sqrt{m/c}(\cos(b_1 t) + i\sin(b_1 t) - \cos(b_2 t) + i\sin(b_2 t)) \tag{4.57}$$

But since $b_1 = -b_2$,

$$u(t) = \frac{1}{2i}\sqrt{m/c}(\cos(b_1 t) + i\sin(b_1 t) - \cos(b_2 t) + i\sin(b_2 t)) \tag{4.58}$$
$$= \sqrt{m/c}\sin\left(\sqrt{c/mt}\right) \tag{4.59}$$

Exactly as we had before

4.3 Complex Eigenvalues

It turns out that eigenvalues can be complex as well and play a very important role. Take a look at the following matrix

$$A = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix} \tag{4.60}$$

The characteristic equation that yields the eigenvalues is

$$(1-\lambda)(3-\lambda) + 5 = 0 \tag{4.61}$$
$$\lambda^2 - 4\lambda + 8 = 0 \tag{4.62}$$
$$\lambda = \frac{4 \pm \sqrt{16-32}}{2} = 2 \pm 2i \tag{4.63}$$

To show the significance of complex eigenvalues, we solve the coupled system of equations:

$$x_1' = x_1 - 5x_2 \tag{4.64}$$
$$x_2' = x_1 + 3x_2 \tag{4.65}$$

which corresponds the matrix formulation

$$\vec{x}' = A\vec{x} \tag{4.66}$$

Let $\lambda_1 = 2 + 2i$ that corresponds to the eigenvector v_1 and $\lambda_2 = 2 - 2i$ that corresponds to the eigenvector v_2. As before, we shall diagonalize matrix A into $V\Lambda V^{-1}$, where

$$\lambda = diag(\lambda_1, \lambda_2) \tag{4.67}$$
$$V = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} \tag{4.68}$$

We can then write

$$V^{-1}\vec{x}' = \Lambda V^{-1}\vec{x} \tag{4.69}$$

Let $\vec{y} = V^{-1}\vec{x}$. We then have the equation

$$y' = \Lambda y \tag{4.70}$$

which is decoupled. Hence, we can write

$$y_1(t) = y_1(0)e^{(2+2i)t} \tag{4.71}$$
$$y_2(t) = y_2(0)e^{(2-2i)t} \tag{4.72}$$

where $y_1(0)$ and $y_2(0)$ are determined by the initial condition (which we did not specify). Then $\vec{x}$ can be written down as

$$x = Vy \tag{4.73}$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{4.74}$$

By specifying the initial conditions and computing the eigenvectors, we can get the exact answers. But instead of doing that, let's examine the solutions of y a bit more

$$y_1(t) = y_1(0)e^{(2+2i)t} \tag{4.75}$$
$$y_2(t) = y_2(0)e^{(2-2i)t} \tag{4.76}$$

What is e^{2+2i} doing to the point $y_1(0)$? Well according to Euler's formula,

$$e^{(2\pm 2i)t} = e^{2t}e^{\pm 2it} \tag{4.77}$$
$$= e^{2t}(\cos(2t) \pm i\sin(2t)) \tag{4.78}$$

e^{2t} represents the magnitude change as a function of t, which in this case is exponentially growing. $\cos(2t) + i\sin(2t)$ corresponds to pure rotation with a frequency of π. Hence, on the phase space, this represents an outward spiral, which is unstable.

On the other hand, imagine if we get an eigenvalue of $-2 \pm 2i$. Then upon exponentiation, it becomes $e^{-2t}e^{\pm 2it}$, whose magnitude is exponentially decreasing. Hence, on the phase space, this corresponds to an inward spiral, which is stable.

We can summarize our findings as follows: for a complex eigenvalue λ out of the matrix A that appears in $\vec{x}' = A\vec{x}$

- The real part of λ controls the stability of the dynamical system: a positive real part means the system is exponentially growing and spiraling outward on the phase plane, indicating instability; a negative real part means the system is exponentially decreasing and spiraling inward on the phase plane, indicating stability
- The imaginary part of λ controls the frequency of rotation of the dynamical system on the phase space.