18.085/18.0851 Comp. Science. Engineering I **Name:** ____________
Summer 2020
Midterm
7/10/2020
Time Limit: 90 Minutes (Self-Enforced) ____________

This exam contains 12 pages (including this cover page) and 5 questions.
Total of points is 60.

You are allowed a sheet of double-side notes. Feel free to write whatever you want. There will be no use of electronic devices

Grade Table (for instructor use only)

Question	Points	Score
1	3	
2	5	
3	10	
4	22	
5	20	
Total:	60	

1. (3 points) Which of the following are suitable Ansatz to the harmonic oscillator,

$$m\frac{d^2u}{dt^2} + cu = 0 \tag{1}$$

where u describes the displacement from equilibrium of a ball of mass m connected to a spring with Hooke's constant c. Circle all that applies

   A. $u(t) = A\cos(bt) + B\sin(bt)$, with $A, B, b \in \mathbb{R}$ to be solved. **CORRECT**

   B. $u(t) = \exp(t)(A\cos(bt) + B\sin(bt))$, with $A, B, b \in \mathbb{R}$ to be solved. **INCORRECT**

   C. $u(t) = \exp(-t)(A\cos(bt) + B\sin(bt))$, with $A, B, b \in \mathbb{R}$ to be solved. **INCORRECT**

2. (5 points) For a square matrix, $A \in \mathbb{R}^{n\times n}$, circle all choices that are equivalent to the statement: A is invertible

   A. The null space of A has only the trivial element, ie. $Nul(A) = \{0\}$. **CORRECT**

   B. The column space of A has dimension n. **CORRECT**

C. A has no repeat eigenvalues. **INCORRECT**

D. The columns of A are linearly independent. **CORRECT**

E. For all $u \in \mathbb{R}^n$, $u^T A u > 0$. **INCORRECT**

3. (10 points) Suppose a matrix $A \in \mathbb{R}^{11\times 375}$ has its singular values following the pattern of $\sigma_i = 10^{12-i}, i = 1, 2, ..., 11$. In other words, the first singular value is 10^{11}, the second is 10^{10}, the third is 10^9, etc.

a) (5 points) What is the condition number of A?

Solution: By definition, the condition number is A is

$$cond(A) = \frac{\text{Largest singular value}}{\text{Smallest singular value}} \tag{2}$$

Since the largest singular value of A is 10^{11} while the smallest singular value is 10, the condition number of A is

$$\begin{aligned} cond(A) &= \frac{10^{11}}{10} && (3) \\ &= 10^{10} && (4) \end{aligned}$$

b) (5 points) Suppose we solve for x in $Ax = b$. Two questions

- i) What are the dimensions of x and b?
- ii) Suppose we do $A\backslash b$ on MATLAB and one of the entries of x is shown on our computer screen as

$$-0.48395748576889907974748658464 \tag{5}$$

Write down the digits you would trust from this answer and explain your reasoning. Feel free to directly cross out the digits shown above

Solution:

i) Since $A \in \mathbb{R}^{11\times 375}$, in order for $Ax = b$ to make sense, $x \in \mathbb{R}^{375}$ and $b \in \mathbb{R}^{11}$.

ii) A general rule of thumb is that in solving $Ax = b$, we can throw away the last $log(cond(A))$ number of digits. Since $\log(cond(A)) = 10$ we need to throw away the last 10 digits. Hence, we can only trust

-0.4839574857688990797~~4748658464~~

4. (22 points) Consider the following boundary value problem over $x \in [0,1]$

$$-u'' = f(x) \tag{6}$$

a) (8 points) Let's take $f(x) = -2\delta(x - 1/4)$ and $u(0) = u(1) = 0$. Solve $u(x)$ and write it in the form of

$$u(x) = \begin{cases} ..., & 0 \leq x < \frac{1}{4} \\ ..., & \frac{1}{4} \leq x \leq 1 \end{cases} \tag{7}$$

Solution:

We shall split $u(x)$ into two parts: u_L for $0 \leq x < \frac{1}{4}$ and u_R for $\frac{1}{4} \leq x \leq 1$. Then since anywhere except at $x = \frac{1}{4}$, $f(x) = 0$, we can write that u_L and u_R linear functions, ie.

$$u_L(x) = Ax + B \tag{8}$$

$$u_R(x) = Cx + D \tag{9}$$

First we apply the natural boundary condition that $u(0) = u(1) = 0$ to conclude that: $B = 0$ and $C = -D$. Hence,

$$u_L(x) = Ax \tag{10}$$

$$u_R(x) = Cx - C \tag{11}$$

Secondly, the continuity at $x = \frac{1}{4}$ gives that

$$A\frac{1}{4} = -\frac{3}{4}C \tag{12}$$

$$A = -3C \tag{13}$$

whence

$$u_L(x) = Ax, \tag{14}$$

$$u_R(x) = -\frac{A}{3}(x - 1) \tag{15}$$

Finally, we apply the integration over a small interval of size 2ϵ around $1/4$

$$\int_{1/4-\epsilon}^{1/4+\epsilon} u'' dx = \int_{1/4-\epsilon}^{1/4+\epsilon} 2\delta(x - 1/4) dx \tag{16}$$

$$u'|_{x=1/4+\epsilon} - u'|_{x=1/4-\epsilon} = 2(1) \tag{17}$$

$$-1/3A - A = 2 \tag{18}$$

$$A = -3/2 \tag{19}$$

Therefore, we can write $u(x)$ as

$$u(x) \begin{cases} -\frac{3}{2}x, & 0 \leq x < \frac{1}{4} \\ \frac{1}{2}(x - 1), & \frac{1}{4} \leq x \leq 1 \end{cases} \tag{20}$$

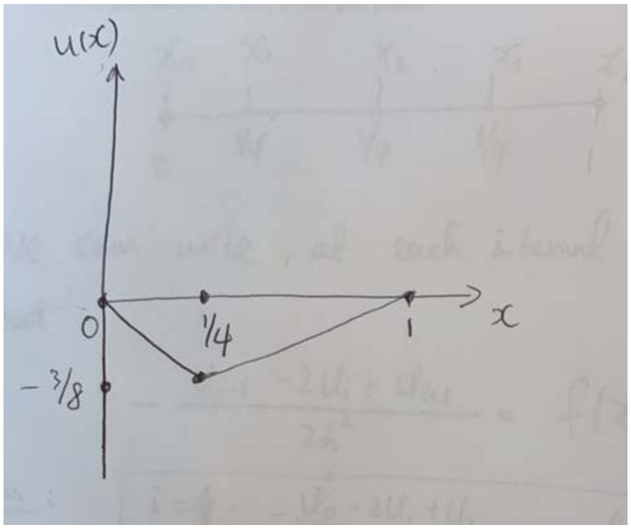

Figure 1: Solution to (4b)

b) (6 points) Sketch the solution including the values of $u(x = 1/4)$

c) (6 points) Let $f = \cos(t)$. Setup but *do not solve* the discretized problem in matrix form $Au = b$ with a grid spacing of $h = 1/4$. The solution vector u to this linear system is our approximation to $u(x)$ at the grid points.

Solution:

We can write, at each interval node $x_1 = 1/4$, $x_2 = 2/4$, $x_3 = 3/4$, that

$$-\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f(x_i) \tag{21}$$

Then for each $i = 1, 2, 3$

$$-\frac{u_0 - 2u_1 + u_2}{h^2} = \cos(1/4) \tag{22}$$
$$-\frac{u_1 - 2u_2 + u_3}{h^2} = \cos(1/2) \tag{23}$$
$$-\frac{u_2 - 2u_3 + u_4}{h^2} = \cos(3/4) \tag{24}$$
$$\tag{25}$$

Note that $u_0 = u_4 = 0$. Hence, we can convert the above three equations into the matrix form $Ax = b$, where

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{26}$$

$$b = \begin{bmatrix} \cos(1/4) \\ \cos(1/2) \\ \cos(3/4) \end{bmatrix} \tag{27}$$

d)(2 points + 2 bonus points) Continuing on with $f = \cos(t)$, if we plot the logarithm of the L^2 error between the analytic and numeric solution against the logarithm of the number of grid points N, we can fit the data to a line

i) (2 points) What is the slope of the line and why?
ii) (2 bonus points) If $\vec{u} = (u_1, .., u_N)$ and $\vec{v} = (v_1, ..., v_N)$ are the numeric and analytic solutions, respectively, evaluated at the grid points $(x_1, ..., x_N)$, can you write down the L^2 error?

Solution:
i) The slope should be -2 since the second difference approximation to the second derivative is second order
ii) The L^2 error is defined as

$$||\vec{u} - \vec{v}|| = \sqrt{\frac{1}{N} \sum_{n=1}^{N} |u_n - v_n|^2} \tag{28}$$

5. (20 points) Suppose $\lambda_1 = 1$ and $\lambda_2 = 2$ are the eigenvalues of a matrix A, and $v_1^T = [1, 0]$ and $v_2^T = [1, 1]$ are the corresponding eigenvectors

a) (4 points) Calculate the matrix A

Using the eigendecomposition formula, we have that

$$A = V\Lambda V^{-1} \tag{29}$$

where

$$\Lambda = diag(\lambda_1, \lambda_2) \tag{30}$$

$$V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{31}$$

$$V^{-1} = \frac{1}{1 * 1 - 1 * 0} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{32}$$

Hence,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \tag{34}$$

b) (4 points) Calculate the matrix A^8, its eigenvalues and its eigenvectors.

Solution:

We recall the property of eigendecomposition and write

$$\begin{aligned}
A^8 &= (V\Lambda V^{-1})^8 && (35)\\
&= V\Lambda^8 V^{-1} && (36)\\
&= \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1^8 & 0\\ 0 & 2^8\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} && (37)\\
&= \begin{bmatrix}1 & 256\\ 0 & 256\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} && (38)\\
&= \begin{bmatrix}1 & 255\\ 0 & 256\end{bmatrix} && (39)
\end{aligned}$$

c) (4 points) For A and A^8, determine if each matrix is positive definite, negative definite, semidefinite, or indefinite

Solution: Since the eigenvalues of A and A^8 are both positive, we know that A and A^8 are positive definite

d) (8 points) Let $u(t) = [u_1(t), u_2(t)]^T$ satisfy

$$u'(t) = Au(t) \tag{40}$$
$$u(0) = [0, 1]^T \tag{41}$$

Solve for $u(t)$.

Solution:

We recall the eigendecomposition $A = V^{-1}$ and plug it into the ordinary differential equation

$$u'(t) = V\Lambda V^{-1}u(t) \tag{42}$$

Let $w(t) = V^{-1}u(t)$. Then

$$w'(t) = \Lambda w(t) \tag{43}$$

which completely decouples the system. We can write down the solution of w as

$$w(t) = \begin{bmatrix} c_1 \exp(\lambda_1 t) \\ c_2 \exp(\lambda_2 t) \end{bmatrix} \tag{44}$$

where c_1 and c_2 are determined by the initial condition. Hence

$$\begin{aligned} u(t) &= Vw(t) && (45)\\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \exp(\lambda_1 t) \\ c_2 \exp(\lambda_2 t) \end{bmatrix} && (46)\\ &= \begin{bmatrix} c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t) \\ c_2 \exp(\lambda_2 t) \end{bmatrix} && (47) \end{aligned}$$

Finally, to find c_1 and c_2, we plug in

$$\begin{aligned} u(t=0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} && (48)\\ &= \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix} && (49) \end{aligned}$$

In this way, $c_2 = 0$ and $c_1 = 1$, whence

$$u(t) = \begin{bmatrix} \exp(t) \\ 0 \end{bmatrix} \tag{50}$$