18.085 Pset #2 Solutions

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Question 1.

The central difference approximation is:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

The taylor expansion of $f(x+h)$ around x is:

$$f(x+h) = f(x) + f'(x)h + f''(x)h^2/2 + f'''(x)h^3/6 + \cdots$$

While the expansion of $f(x-h)$ has negative terms for odd orders of h:

$$f(x-h) = f(x) - f'(x)h + f''(x)h^2/2 - f'''(x)h^3/6 + \cdots$$

Then we can see the approximation differs from the actual $f'(x)$ by terms of h^2 or smaller:

$$\begin{aligned}\frac{f(x+h) - f(x-h)}{2h} &= \frac{2f'(x)h + 2f'''(x)h^3/6 + \cdots}{2h} \\ &= f'(x) + f'''(x)h^2/6 + \cdots = f'(x) + \mathcal{O}(h^2)\end{aligned}$$

hence it is indeed second order. $\mathcal{O}(h^2)$ is notation for functions of h that are upper bounded by a constant multiple of h^2: Wikipedia article.

Question 2.

Part 1: Integrate twice to find $u(x)$:

$$\begin{aligned}u'(x) &= \int u''(x)dx = \int -\cos(2x)dx = -\frac{1}{2}\sin(2x) + C \\ u(x) &= \int u'(x)dx = \int \left(-\frac{1}{2}\sin(2x) + C\right) dx = \frac{1}{4}\cos(2x) + Cx + D\end{aligned}$$

By the initial conditions:

$$u(0) = 3 = \frac{1}{4} + D \implies D = \frac{11}{4}$$
$$u(1) = 3 = \frac{1}{4}\cos(2) + C + \frac{11}{4} \implies C = \frac{1}{4} - \frac{1}{4}\cos(2)$$

Part 2: See solution MATLAB code.

Question 3.

The integral of the delta function $\delta(x)$ is the step function $H(x)$, so we get:

$$\begin{aligned} u'(x) = \int u''(x)dx &= -\int \delta(x - 1/4) + 4\int \delta(x - 1/2) \\ &= -H(x - 1/4) + 4H(x - 1/2) + C \end{aligned}$$

Integrating once more, knowing that the integral of $H(x)$ is a "ramp" function which can be written as $\max(0, x)$:

$$u(x) = \int u'(x)dx = -\max(0, x - 1/4) + 4\max(0, x - 1/2) + Cx + D$$

Then determine C and D with the initial conditions:

$$\begin{aligned} 0 = u(0) = -\max(0, -1/4) + 4\max(0, -1/2) + D & \\ &\implies D = 0 \\ 0 = u(1) = -\max(0, 3/4) + 4\max(0, 1/2) + C & \\ = -3/4 + 2 + C & \\ &\implies C = -5/4 \end{aligned}$$

The solution can be written on three intervals as:

$$u(x) = \begin{cases} -5x/4, & \text{for } 0 \le x \le 1/4 \\ -9x/4 + 1/4 & \text{for } 1/4 \le x \le 1/2 \\ 7x/4 - 7/4 & \text{for } 1/2 \le x \le 1 \end{cases}$$

Question 4.

A symmetric 2×2 matrix has the general form:

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

For this to be an orthogonal matrix, we require the columns to be orthogonal, so $ab + bc = b(a + c) = 0$. Then note that either $b = 0$ or $a + c = 0$ but not both, as the zero matrix is not orthogonal. In the case of $b = 0$, then $\mathbf{A}$ is diagonal and the eigenvalues are just a and c. From the orthogonality condition we further have:

$$\mathbf{A}\mathbf{A}^T = \begin{pmatrix} a^2 & 0 \\ 0 & c^2 \end{pmatrix} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $a, c = \pm 1$ and so the eigenvalues can either be 1 or -1. In the case of $a + c = 0$ we also require $a^2 + b^2 = 1$ for each column to be a unit vector. Hence the matrix is:

$$\mathbf{A} = \begin{pmatrix} a & \sqrt{1 - a^2} \\ \sqrt{1 - a^2} & -a \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

The θ parametrization is evident from $a^2 + b^2 = 1$ being satisfied by the identity $\cos^2\theta + \sin^2\theta = 1$. The characteristic polynomial for this matrix is $(\cos\theta - \lambda)(-\cos\theta - \lambda) - \sin^2\theta = \lambda^2 - \cos^2\theta - \sin^2\theta = \lambda^2 - 1$, which has roots -1 and 1. Thus in both cases the possible eigenvalues are ± 1.

For a general proof of eigenvalues being ± 1, given an $n \times n$ orthogonal symmetric matrix $\mathbf{A}$ there is an eigendecomposition $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$. Orthogonality and symmetry of $\mathbf{A}$ means $\mathbf{A}\mathbf{A}^T = \mathbf{A}^2 = \mathbf{I}$ so:

$$\mathbf{A}\mathbf{A}^T = (\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1})(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}) = \mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^{-1} = \mathbf{I}$$

$$\mathbf{V}\mathbf{\Lambda}^2 = \mathbf{V} \implies \mathbf{\Lambda}^2 = \mathbf{I}$$

Then clearly the eigenvalue entries λ_i of $\mathbf{\Lambda}$ must satisfy $\lambda_i^2 = 1$ so $\lambda_i = \pm 1$.

Question 5.

Part 1:

Our first system is:

$$\begin{cases} u' = 6u - v \\ v' = 2u + 3v \\ u(0,0) = (1,-1) \end{cases}$$

In matrix form:

$$\mathbf{w}' = \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{A}\mathbf{w}$$

The characteristic polynomial of $\mathbf{A}$ is $(6-\lambda)(3-\lambda)+2$ which has roots $\lambda_1 = 4$ and $\lambda_2 = 5$. The two eigenvectors can then be found to be $\mathbf{u}_1 = (1,2)^T$ and $\mathbf{u}_2 = (1,1)^T$. Then the general solution to the diffeq system is:

$$\mathbf{w}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2$$

All that is left is to find c_1 and c_2. At $t = 0$ we have

$$\mathbf{w}(0) = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

From which we can solve $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ to get solution:

$$\mathbf{w}(t) = -2e^{\lambda_1 t}\mathbf{u}_1 + 3e^{\lambda_2 t}\mathbf{u}_2 = -2e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Part 2:

The second system is:

$$\begin{cases} u' = u + 2v \\ v' = 3u + v \\ u(0,0) = (1,-1) \end{cases}$$

In matrix form:

$$\mathbf{w}' = \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{A}\mathbf{w}$$

The characteristic polynomial of $\mathbf{A}$ is $(1-\lambda)(1-\lambda)-6$ which has roots $\lambda_1 = 1-\sqrt{6}, \lambda_2 = 1+\sqrt{6}$. The two eigenvectors can then be found to be $\mathbf{u}_1 = (\sqrt{2/3}, 1)^T$ and $\mathbf{u}_2 = (-\sqrt{2/3}, 1)^T$. The general solution is

$$\mathbf{w}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2$$

At $t = 0$ we have:

$$\mathbf{w}(0) = c_1 \begin{pmatrix} \sqrt{2/3} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -\sqrt{2/3} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2/3} & -\sqrt{2/3} \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 11 \end{pmatrix} = \begin{pmatrix} 11/2 + \sqrt{3/8} \\ 11/2 - \sqrt{3/8} \end{pmatrix}$$

See next pages for phase portraits:

In [1]:
```julia
using PyPlot
using LinearAlgebra
```

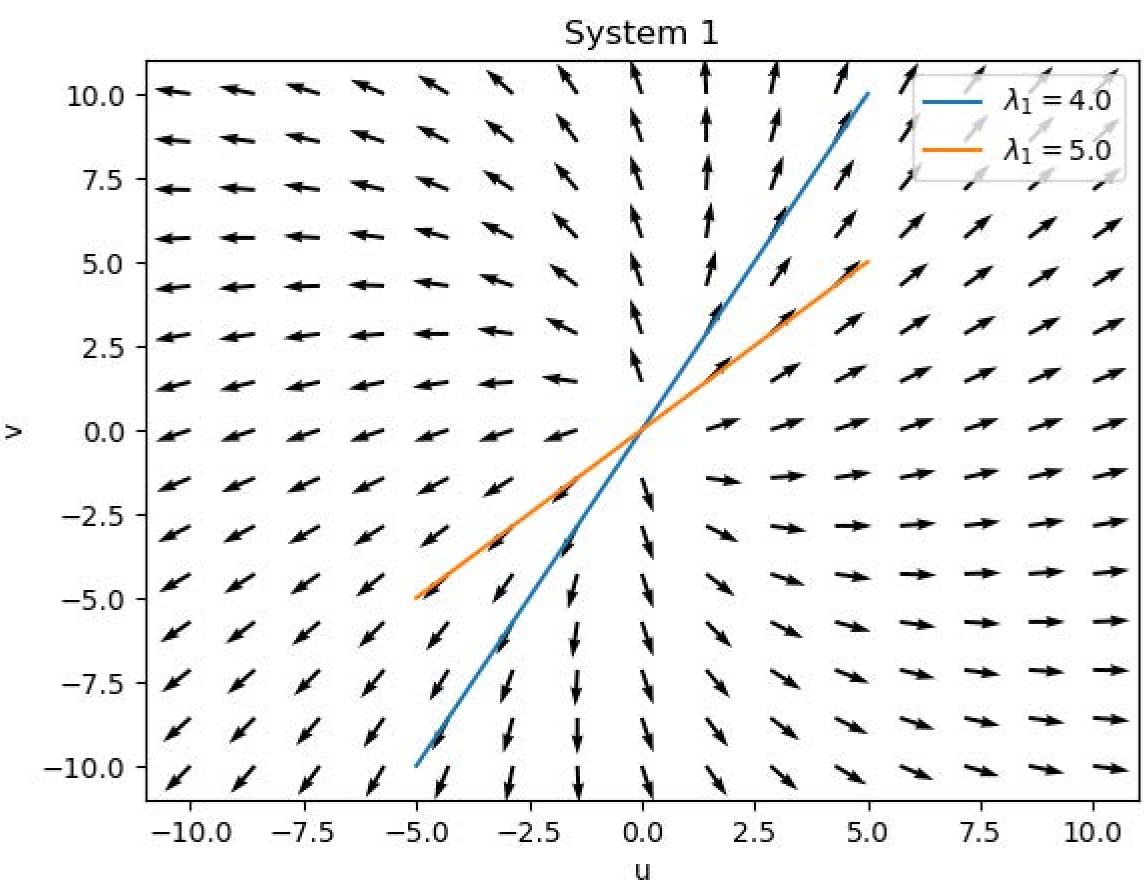
In [2]:
```julia
function plotsystem(A, name)
    lam = eigen(A).values
    V = eigen(A).vectors

    axis = range(-10, 10, length=15)
    meshgrid(x, y) = (repeat(x, outer=length(y)), repeat(y, inner=length(x)))
    x_ax, y_ax = meshgrid(axis, axis)
    UVs = A * [x_ax y_ax]'
    UVs ./= sqrt.(sum(UVs.^2, dims=1))
    u, v = UVs[1,:], UVs[2,:]

    quiver(x_ax, y_ax, u, v)
    V ./= V[1,:]'
    x_ax = range(-5, 5, length=15)
    plot(x_ax, x_ax * V[2,1], label="\$\\lambda_1 = $(lam[1]) \$")
    plot(x_ax, x_ax * V[2,2], label="\$\\lambda_1 = $(lam[2]) \$")
    xlabel("u"), ylabel("v")
    legend()
    title(name)
end
```

Out[2]:
```
plotsystem (generic function with 1 method)
```

In [3]:
```julia
A₁ = [6 -1 ; 2 3]
plotsystem(A₁, "System 1");
```



In [4]:

```
A₂ = [1 2 ; 3 1]
plotsystem(A₂, "System 2");
```