18.091 Introduction to Dynamical Systems Spring 2005

Lecture 11: Fractals and Dimension

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Note: the spaces considered in this lecture are all metric spaces.

Fractals might appear at first to be unrelated to our current studies. However, they are connected to dynamical systems in an interesting way: a number of dynamical systems have orbits that approach a set which is itself a fractal. This portion of the lecture will cover the definition of a fractal and a few examples of such.

Definition A *fractal* is a subset of $\mathbf{R}^n$ that is self-similar and whose fractal dimension exceeds its topological dimension.

Now we need to discuss the concepts that are involved in this definition. The first, self-similarity, is something we have seen before in dynamical systems.

Definition A subset of $\mathbf{R}^n$ is *(affine) self-similar* if a subset of this subset is mapped to the original subset by a non-trivial affine transformation : $f(x) = \mathbf{A}x + \mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ invertible matrix and $\mathbf{b}$ is an n dimensional vector. The transformation is known as a self-similarity transformation.

Example We can construct a region that is explicitly self-similar via a simple region and an affine transformation. Let $R \subset \mathbf{R}^2$ be as follows. Define $S = [-1, 1] \times [-1, 1]$. Let $F(x) : \mathbf{R}^2 \to \mathbf{R}^2$ be $F(x) = (1/2)x + (1, 0)$. Then $R = \cup_{i=-\infty}^{\infty} F^i(S)$ is self-similar and part of it can be seen below.

Example The orbit diagram for the quadratic map is self similar in the sense that magnifying certain regions (which means recentering and then scaling) produces an image that looks the same as the original. The recentering and then scaling can be implemented via an affine transformation.

The next concept that we need to define is that of fractal dimension. However, first we need to define a more intuitive concept, that of topological dimension. Topological dimension is a concept meant to correspond to our intuitive notion of "the number of independent ways one can move within an object". We intuitively think of a line as one dimensional because there's only one independent way one can move on a line, and similarly, a plane would be two dimensional.

We define topological dimension as an inductive concept. First we have the base case:

Definition A set S has *topological dimension* if every point has arbitrarily small neighborhoods, that is, neighborhoods U with $sup(d(x, y)|x, y \in U)$ arbitrarily small whose boundaries do not intersect the set.

Theorem Every connected component of a non-empty set of topological dimension zero is a point.

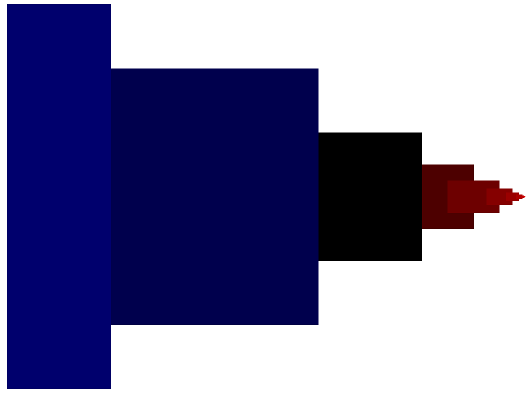

Figure 1: The self similar region R.

Proof Say a connected component C of a set of topological dimension zero contains two distinct points, call them p and q, and let $d \equiv d(p,q)$. Given an open set U containing p such that $sup(d(x,y)|x,y \in U) < d$. Then U does not contain q, and thus the boundary U must intersect C or else U and the interior of U^c separate C, which is impossible since C is connected. But then the set is not of topological dimension zero. Hence C has either one or zero points.

Definition A set S has *topological dimension* k if each point in S has arbitrarily small neighborhoods whose boundaries meet S in a set of dimension $k-1$, and k is the least nonnegative integer for which this holds.

Example A line in a higher dimensional space has topological dimension 1, since p contained in the line implies that $B(p,r)$'s boundary intersects the line in two disjoint points, which comprise a set of topological dimension 0. The line is connected, and so is not of topological dimension 0.

However, as we will see, the topological dimension of objects is not sensitive enough of a measure to describe the intrinsic properties of fractals. In fact, using the notion of topological dimension, we essentially get a "lower bound" for the dimension of a set. In each of the following pictured sets, it seems as if the topological dimensions do not fully capture the nature of the sets:

We can embellish our notion of self-similarity from before. In the definition of self-similarity, we singled out a subset of a self-similar set. If it is possible to divide a set into several disjoint subsets, each of which is conrguent (can be mapped onto another such set by means of a distance preserving transformation) to all of the others, then we can associate two numbers with a particular division of the set into regions. The first is the number of regions that the original region has been divided into. To get the second number, we must further restrict the set of self-similarity transformations that are allowed to those for which the matrix $\mathbf{A}$ is a constant times the identity matrix. Then we can also associate a "magnification factor" with such a transformation, which is this constant.

Given a self-similar set, we define the *fractal dimension* D of this set as $\frac{\ln k}{\ln M}$ where k is the number of disjoint regions that the set can be divided into, and M is the magnification factor of the self-similarity transformation.

Now we see that the box fractal, Sierpinski triangle, and Koch curve, which is defined as the

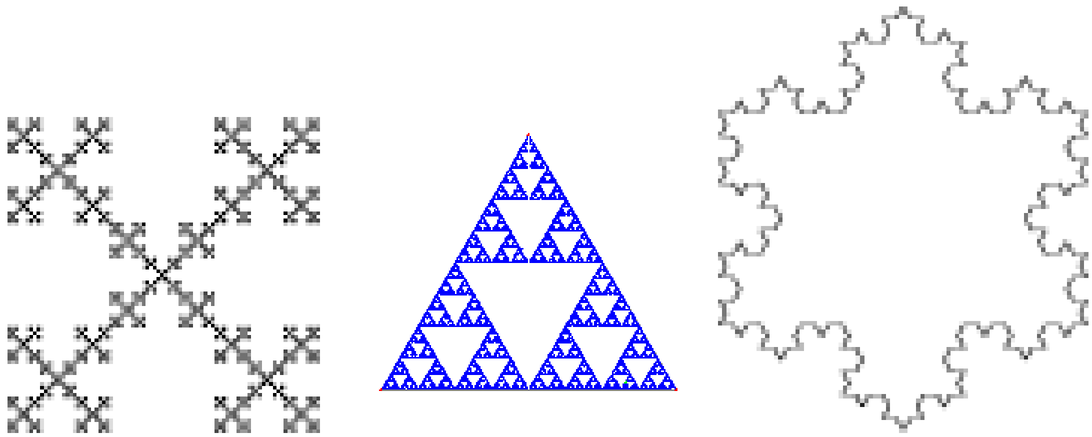

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Figure 2: The box fractal and Sierpinski triangle each have topological dimension 1, and the Koch snowflake has topological dimension 0, but all these seem intuitively "bigger" than their topological dimensions indicate. (Figures are displayed from left to right)

intersection of the Koch snowflake with a $\frac{2\pi}{3}$ radian region of the plane centered about the y-axis, are all fractals:

The box fractal is created by dividing a square into nine identical regions and selecting five of them in a particular way, and then repeating this on the newly produced squares. Thus we have that at a given stage the magnification factor is 3 and the number of regions is 5, leading to $D = \frac{\ln 5}{\ln 3} = 1.46497\ldots$. However, one might wonder if the fractal dimension is a well defined concept, considering that we can also divide the fractal into 25 self-similar regions, each of which needs a transformation with magnification 9 to be mapped to the original set. However, we notice that $25 = 5^2$ and $9 = 3^2$, which means that the fractal dimension is so far well-defined. While not so easy to prove, it is easy to see that any division of the region into a number of pieces each of which is self-similar to the original will produce the same fractal dimension, since the number of pieces will be 5^m and the magnification factor will be 3^m for such a division.

Similarly, we see that the Koch curve has a fractal dimension of $\frac{\ln 4}{\ln 3} = 1.26186\ldots$, and the Sierpinski triangle has fractal dimension $\frac{\ln 3}{\ln 2} = 1.58496\ldots$.