18.091 Lecture 11
Section 14.7: Iterated Function Systems

Laura Hajj

May 6, 2005

Now we turn to iterated function systems(IFS), which are used to create and analyse fractals.

Definition. Let $0 < \beta < 1$ and let $p_1, ...p_n$ be points in the plane. Let $A_i(p) = \beta(p - p_i) + p_i$ for all $i = 1...n$. The collection of all functions $A_1, ...A_n$ is called an *iterated function system*.

The IFS draws orbits into the fixed point p_i with β as the contraction factor. We use the IFS to produce a fractal by using randomized iteration of the A_i on some arbitrary initial conditions.

Example. The Chaos Game IFS produces the Sierpinski Triangle Fractal

Arbitrarily choose a fixed point

$$p = \begin{pmatrix} x \\ y \end{pmatrix}$$

and suppose

$$p_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, i = 1, 2, 3$$

are the vertices of an equilateral triangle. Then

$$A_i = 0.5|p - p_i| + p_i$$

is the IFS for the chaos game and produces the Sierpinski Triangle. Note that β here is the invere of the magnification factor showing the self-similarity of the Triangle.

Definition. The set of points to which an arbitrary obit in the plane converges is the *attractor* for a given iterated function system.

It can be proven that an attractor exists for every IFS; however, proof is omitted here.

Example. The Cantor Middle-Thirds Set as Attractor

I claim that the Cantor middle-thirds set is an attractor for the following IFS:

$$\begin{aligned} A_0 \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} x \\ y \end{pmatrix} \\ A_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} x-1 \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

By observation, $y_n = \frac{y_0}{/} 3^n$ regardless of the order chosen for iteration of these two contraction functions. So, the orbit of any y_0 tends to $y = 0$ at a geometric rate. To examine the behaviour of the x's after iteration, we will compute the orbit of x_0 by alternating between using function A_0 and A_1. That is, the sequence of iterations is $\{A_0, A_1, A_0, A_1..\}$ Starting with A_0, our alternating iteration yields

$$x_n = \frac{x_0}{3^n} + \left(\frac{2s_1}{3^n} + \frac{2s_2}{3^{n-1}} + ... + \frac{2s_n}{3} \right),$$

where $s_i = 0$ when i is odd and $s_i = 1$ when i is even.

As $n \to \infty$, $x_n \to \sum_{i=0}^{\infty} \frac{t_i}{3^i}$ where t_i are alternately 0 or 2. Consequently, the sequence of x_n converges to one value if n is even and another if n is odd:

$$\lim_{n \to \infty} (x_{2n}) = \sum_{i=0}^{\infty} \frac{2}{3^{2i-1}} = \frac{3}{4}$$

$$\lim_{n \to \infty} (x_{2n+1}) = \sum_{i=0}^{\infty} \frac{2}{3^{2i}} = \frac{1}{4}.$$

We saw both of these points as elements of the Cantor middle-thirds set in Section 7.3. This argument is easily expanded to cover general case, which then satisfies my claim. So, all orbits of this IFS converge to the Cantor middle-thirds set. Notice, however, that this orbit does not converge to a mere point of the Cantor set but eventually visits all regions upon sufficient randomized iterations of A_0 and A_1.

Many variations on the idea of the iterated function system exist. For example, linear contractions may contract points unequally or with different probabilities. We will now consider an IFS that not only contacts points towards but also rotates points around a fixed point.

Example. Contraction and Rotation

For our IFS, we construct a function, which contracts points by β and rotates points by θ:

$$A\begin{pmatrix}x\\y\end{pmatrix} = \beta\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\cdot\begin{pmatrix}x-x_0\\y-y_0\end{pmatrix}+\begin{pmatrix}x_0\\y_0\end{pmatrix}$$

Choosing $\beta = 0.9$, $\theta = \frac{\pi}{2}$, and fixed point $p_0 = \begin{pmatrix}1\\1\end{pmatrix}$, we get

$$\begin{aligned} A\begin{pmatrix}x\\y\end{pmatrix} &= 0.9\begin{pmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix}\cdot\begin{pmatrix}x-1\\y-1\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}\\ &= 0.9\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}x-1\\y-1\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}\end{aligned}$$

The graph of this function is on page 196 of our text.