18.091 Lecture 7
Chapter 10: Chaos

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1 Introduction

Chapter 10 finally introduces chaos and offers a definition of chaotic dynamical systems. Despite the complexity of chaotic dynamical systems, many chaotic systems can be fully understood through semiconjugacy, which we will also discuss. Today, we will explore the properties of chaos by examining the shift map and the quadratic map with constant -2 as examples.

2 Properties of Chaotic Systems and the Shift Map

Chaos has many possible definitions and the scientific community has reached no consensus as to which definition is most accurate. This definition was chosen for its simplicity and the ease of its use for verifying systems as chaotic.

Definition: A dynamical system F is *chaotic* if:

1. Periodic points for F are dense.
2. F is transitive.
3. F depends sensitively on initial conditions.

In fact, the first two definitional requirements imply the third; although, a proof of this claim is left to the reader.

1. Density is a topological idea which states that if S is dense in X then for each $x \in X$ there is a point $s \in S$ arbitrarily close to x. Rigorously,

Definition: Let X be a metric space. A subset S of X, is *dense in* X if every point of X is a limit point of S, or a point of S.

So, for any point f in a chaotic system F one can find a periodic point $f_n \in F$ such that $d[f, f_n] < \epsilon$ for all $\epsilon > 0$.

2. Transitivity expands the idea of arbitrary closeness to orbitals by requiring that for any two points in a transitive system there exists an orbit that comes arbitrarily close to both points. That is,

Definition: A dynamical system is *transitive* if for any two points x, y in the system's space and any $\epsilon > 0$ there exists a third point z where $d[z, x] \leq \epsilon$ that has an orbital which comes within ϵ of y.

We observe that any dynamical system with a dense orbit is also transitive because a dense orbit passes arbitrarily close to every point in the system. The converse of this statement, the Baire Category Theorem, is also true and proof is left as an exersize to the interested student.

3. Sensitive Dependence to Initial Conditions requires that for any x in the system there are points arbitrarily close to x with orbits that are eventually far away from x. More formally,

Definition: A dynamical system F is *sensitively dependent on initial conditions* if there is a $\beta > 0$ such that for any $x \in F$ and $\epsilon > 0$ there is a y such that $d(x, y) \leq \epsilon$ and there is a k such that $d[F^k(x), F^k(y)] \geq \beta$.

Sensitive dependence on initial conditions causes many problems for applied mathematician because the divergence of orbits may cause even the most accurately measured initial conditions to produce an orbit dramatically different from the actual initial condition.

Finally, we fully understand the definition of chaotic dynamical systems. In short, a chaotic map is unpredictable, indecomposable, and yet somewhat regular due respectively to sensitivity, transitivity, and density.

Example: The iteration of the Shift Function on the Sequence Space is chaotic.

We will examine the three definitional requirements in detail by using them to directly verify that the iterated shift function σ on the sequence space Σ as an example of a chaotic system. Recall that the sequence space Σ is defined by

$$\Sigma = \{(s_0 s_1 s_2 ...) | s_j = 0 \text{ or } 1\}.$$

Also, the definition of the shift function $\sigma : \Sigma \to \Sigma$ is:

$$\sigma(s_0 s_1 s_2 s_3 ...) = (s_1 s_2 s_3 ...).$$

First, will prove that the subset $P \subset \Sigma$ containing all periodic points for σ is dense in Σ. To do this, we must simply show that for any $\mathbf{s} \in \Sigma$ there exists a point $\mathbf{p} \in P$ such that $d[p, s] \leq \epsilon$ for all ϵ. We will do this using the Proximity Theorem from last class.

Proximity Theorem: Let $\mathbf{t}$,$\mathbf{u} \in \Sigma$. Then $t_i = u_i$ for $i = 0, 1, ..., n$ if and only if $d[\mathbf{t},\mathbf{u}] \leq 1/2^n$.

Suppose $\epsilon > 0$ is given. Choosing an arbitrary $\mathbf{s} \in \Sigma$, we can construct a point $\mathbf{p_n} = (s_0 s_1 s_2 ... s_n \overline{s_0 s_1 s_2 ... s_n})$, which shares its first $n+1$ entries with $\mathbf{s}$, that has period $n+1$ for σ. So, choosing a large enough n and applying the Proximity Theorem, we find

$$d[\mathbf{s}, \mathbf{p_n}] \leq 1/2^n < \epsilon.$$

Now, we will prove that the shift map is transitive by using our observation that a dense orbital implies transitivity. Given an $\mathbf{s} \in \Sigma$ and $\epsilon > 0$, we again choose a n such that $1/2^n < \epsilon$. Then, using the Proximity Theorem, we can find a point $s \in \Sigma$, whose first $n+1$ entries are identical to $\mathbf{s}$, that satisfies

$$d[\sigma^k(s), \mathbf{s}] \leq 1/2^n < \epsilon$$

for some integer k. This equation is satisfied by the point $\hat{s}$, which consists of all possible 1-digit permutations of 0's and 1's in a block, followed by all 2-digit permutations in a block, and so on. That is, $\hat{s} = (0\,1\,00\,01\,10\,11...)$. Regardless of $\mathbf{s}$, $\hat{s}$ contains the first $n+1$ entries, $(s_0 ... s_n)$, of $\mathbf{s}$ as one of the permutations of the block of length $n+1$. Suppose the sequence $(s_0 ... s_n)$ begins at the j entry of $\hat{s}$ then we can satisfy the above equation by choosing $k = j$. Thus, we find that the orbit around $\hat{s}$ of σ comes arbitrarily close to all points in Σ.

Finally, we will show that the Shift Map is sensitively dependent on initial conditions. We will do this by showing that $\beta = 1$ satisfies this definitional requirement. Given an $\mathbf{s} \in \Sigma$ and an $\epsilon > 0$, we again choose an large enough n to satisfy $1/2^n < \epsilon$. Application of the Proximity Theorem shows that for any $\mathbf{t} \neq \mathbf{s}$ such that $d[\mathbf{s}, \mathbf{t}] \leq 1/2^n < \epsilon$, $d[s_k, t_k] = |s_k - t_k| = 1$ for at least one integer $k > n$. Therefore, $d[\sigma^k(\mathbf{s}), \sigma^k(\mathbf{t})] \geq d[s_k, t_k] = 1 = \beta$.

3 Semiconjugacy and the Quadratic Map

Although we easily directly verified that $\sigma : \Sigma \to \Sigma$ is a chaotic dynamical system, many systems are much more difficult to analyze, like the quadratic map. We may try to use conjugacies to compare complicated systems to other more easily understood and thereby indirectly establish chaos. However, the one-to-one requirement for conjugacies is often restrictive and unnecessary for proving that a system is chaotic by comparison. Instead, we will use semiconjugacies to convert the orbits of one system into the orbits of another system.

Definition: Suppose $A : X \to X$ and $B : Y \to Y$ are two dynamical systems. A mapping $h : X \to Y$ is a *semiconjugacy* if h is continuous, onto, at most n-to-one, and satisfies

$$h \circ A = B \circ h.$$

Our analysis of the quadratic map will elucidate this concept but before we move on to our example we need to establish one more prerequisite proposition.

Proposition 1 (The Density Proposition) *Suppose $F : X \to Y$ is a continuous map that is onto and suppose also that $D \subset X$ is a dense subset. Then $F(D)$ is dense in Y.*

Proof Given an arbitrary $y_0 \in Y$ and $\epsilon > 0$, we must find a point $z \in F(D)$ such that $d[y_0, z] \leq \epsilon$. Since F is onto there exists an $x_0 \in X$ such that $F(x_0) = y_0$. The density of D in X implies that we choose an $\hat{x} \in D$ such that $d[\hat{x}, x_0] \leq \delta$ for all $\delta > 0$. The continuity of F means that we there exists a $\delta > 0$ such that $d[F(x), F(\hat{x_0})] \leq \epsilon$. Setting $z = \hat{x}$, we find that $d[y_0, z] = d[F(x_0), F(\hat{x})] \leq \epsilon$.

Example: The Quadratic Function $Q_{-2}(x) = x^2 - 2$ is chaotic on $[-2, 2]$.

We will explore this example by comparing Q_{-2} to the function $V(x) = 2|x| - 2$. First, we will verify that $V(x)$ is chaotic.

1. The graph of V^n is 2^n straight lines, each with a slope of $\pm 2^n$ and a domain interval of length $1/2^{n+2}$. The iterated function $V^n(x)$ has fixed points in each subinterval $I \subset [-2, 2]$ of length $1/2^{n+2}$ for all n. Since periodic points of $V^n(x)$ cycle among fixed points for $V^n(x)$ and n can be made arbitrarily large, the periodic points are dense in $[-2, 2]$.

2. Given $x \in I$, $y \in [-2, 2]$, and $\epsilon > 0$, we can choose an n such that the length of $I < \epsilon$. Since the $V^n(I)$ maps onto $[-2, 2]$ there exists a $z \in I$ which maps to y, which implies that V has a dense orbit and is therefore transitive on $[-2, 2]$.

3. For every $x \in I$ there is a $y \in I$ such that $|V^n(x) - V^n(y)| \geq 2$, because each I maps onto $[-2, 2]$. Choosing $\beta = 2$, we see that $V(x)$ is sensitively dependent on initial conditions.

To prove that $Q_{-2}(x) = x^2 - 2$ is chaotic we want to show that $Q_{-2}(x)$ is dynamically equivalent to $V^n(x)$. Consider the function $C = -2cos((\pi x)/2)$, which maps $[-2,2]$ onto itself, is at most 2-to-1 on $[-2,2]$, and is continuous by inspection. Although, C cannot be a homeomorphism between V and Q_{-2}, it does carry orbits of V to orbits of Q_{-2}. Since C is continuous and onto, the Density Proposition yields that Q_{-2}'s periodic points are dense in Q_{-2} and that there exists a dense orbit for Q_{-2}. Because V^n maps any interval of length $1/(2^{n-2}) \subset [-2,2]$ onto $[-2,2]$ and C carries orbits of V to Q_{-2}, Q_{-2} must do the same, which shows that Q_{-2} is sensitively dependent on initial conditions.